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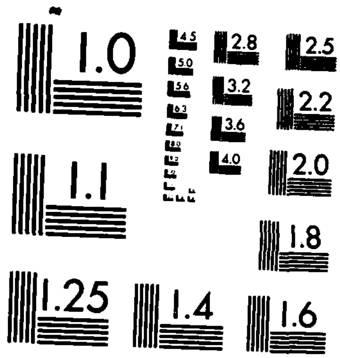

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THERMOMECHANICAL RESPONSE OF SHELLS WITH EMPHASIS ON A CONICAL SHELL

Final Report

December 1984

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This research is concerned with the thermomechanical response of shells, with emphasis on a conical shell. Four topics of interest were identified and the results for each of these were written in the form of journal articles and submitted for publication.

Topic 1 was concerned with developing a uniqueness theorem for thermoelastic shells that admits generalized boundary conditions. Topic 2 was concerned with developing a nonlinear constrained theory of shells that includes tangential shear deformation. Topic 3 entailed proposing new values for certain constitutive coefficients for shells. Finally, we focused on Topic 4, which was concerned with heat conduction in rigid plates and shells, with emphasis on a conical shell. For each of these topics, we modeled the shell as a Cosserat surface.

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19. ABSTRACT (Continued)

The conical shell was of particular interest because it has a converging geometry such that the shell near its tip is necessarily "thick" even though the shell near its base may be "thin." To develop confidence in the Cosserat theory for both the thin-shell and thick-shell limits, we considered a number of problems for plates, circular cylindrical shells, spherical shells, and a conical shell. It was shown that by appropriately modifying the constitutive equations, it is possible to include enough geometrical features of the shell to predict relatively accurate results even in the thick-shell limit.

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SUMMARY

A topic of ongoing interest to the Air Force is the response of shell structures to thermal loads. These thermal loads may be caused by aerodynamic heating on aircraft and missiles, by laser weapons, or by radiation effects on space structures. In all cases, it is desirable to determine the thermomechanical response of shell-like bodies that model various components of aircraft structures.

To determine the complete thermomechanical response of a shell-like body to thermal loads, it is necessary to calculate both the thermal and mechanical fields. The problems with analytically solving the three-dimensional thermal and mechanical equations for general structural geometry and general loading are currently insurmountable. Therefore, alternative methods of solution are required. One alternative is to obtain a numerical solution using large finite element codes. Although obtaining a numerical solution of this kind is within current technological capability, there are two important disadvantages with this method. First, it may be necessary to use many elements and many time steps to calculate the solution of a dynamic problem. This means that use of the method as a design tool when a number of parameters must be varied would be prohibitively expensive. Second, the method necessarily calculates many details of the thermal and mechanical fields that are not of particular interest. For example, for shells it is sufficient to obtain limited information such as the resultant force and moment applied to the shell. Consequently, the details of the stress distribution through the thickness of the shell are not needed.

In this research, we used an alternative method of solution that judiciously models the structure by an appropriate shell theory having equations that are considerably simpler than those of the three-dimensional theory. Such a theory focuses attention solely on quantities of primary interest. Consequently, even if it is necessary

to develop a numerical solution of the shell equations, the computer time is efficiently used, the results are easily interpreted, and the method can be used for design purposes.

The objective of this research was to analyze the thermomechanical response of a conical shell (Figure 1) that is a model for aircraft and missile nose cones. Because the conical shell has a converging geometry, the shell near its tip is necessarily "thick" even though the shell near its base may be "thin." Therefore, it is important to model the conical shell with a theory that accurately incorporates the geometrical details of this crucial tip region. We used recent developments in the Cosserat theory for the thermomechanical response of shells. We based our development on the Cosserat theory mainly because this theory accurately models the geometry of the shell and is sufficiently general enough to include the important effects of the steep temperature profile through the thickness of the shell (e.g., tangential shear deformation,* higher order displacement and temperature profiles through the thickness of the shell) without the complexity of the complete three-dimensional theory.

This research was divided into four parts, each of which has been written as a technical paper that has been submitted for publication. A copy of each paper is included as an appendix:

Appendix A: A Uniqueness Theorem for Thermoelastic Shells with Generalized Boundary Conditions

Appendix B: A Nonlinear Constrained Theory of Shells that Includes Tangential Shear Deformation

Appendix C: On the Determination of Certain Constitutive Coefficients for Thermoelastic Shells

Appendix D: Heat Conduction in Plates and Shells with Emphasis on a Conical Shell.

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*We use the term tangential shear deformation instead of the usual term "transverse" shear deformation because it is more descriptive when considering nonlinear deformation of a shell as opposed to linear deformation of a plate.

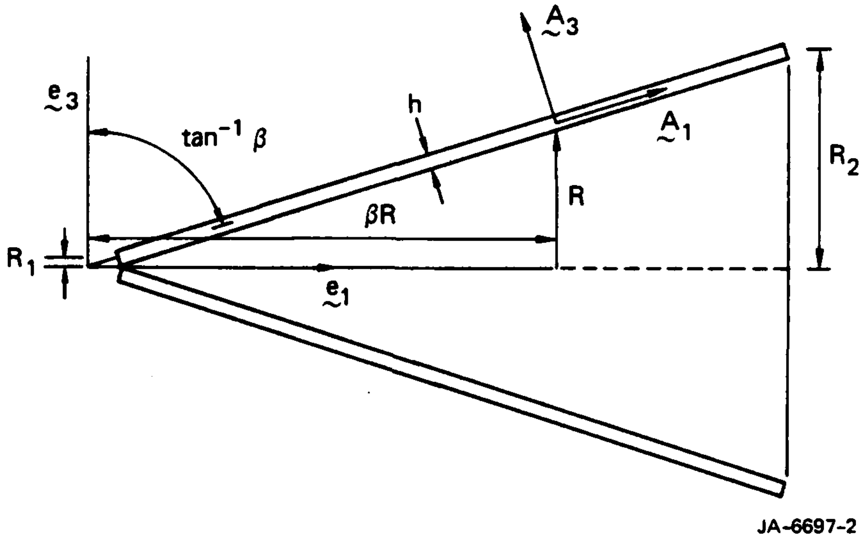

FIGURE 1 CONICAL SHELL OF CONSTANT THICKNESS h, TIP RADIUS R_1, AND BASE RADIUS R_2

Basically, Appendices A-C deal with certain theoretical issues that were observed while developing the general equations to analyze the thermomechanical response of shells, whereas Appendix D deals with the main issue of the conical shell. Appendix A entails a generalization of the uniqueness theorem for linear thermoelastic shells to include mechanical contact with a linear elastic medium and thermal convection on both the major surfaces of the shell and its boundary curve. Appendix B is concerned with developing a nonlinear constrained theory of shells that includes tangential shear deformation. This topic was considered because tangential shear deformation is expected to be important when a shell is loaded with a steep through-the-thickness temperature gradient. Appendix C is concerned with determining certain thermal and mechanical constitutive coefficients for thermoelastic shells by direct comparison with exact solutions. It was observed that better comparison with simple exact solutions could be obtained by specifying values for certain coefficients that are different from those previously proposed.

In addition to considering general aspects of thermomechanical response of shells, in Appendix D, we considered heat conduction in rigid shells with particular emphasis on a conical shell. Specifically, we used shell equations based on the theory of a Cosserat surface to determine the average (through the thickness) temperature and temperature gradient in rigid shells. Attention was focused on rigid shells because when the strain rates in a deformable shell are small enough, the thermal and mechanical problems uncouple in the sense that the temperature fields can be determined from equations for heat conduction in rigid shells. Once these temperature fields are known, they can be used together with constitutive equations to calculate thermal loads which cause mechanical deformation.

To develop confidence in the Cosserat theory for both the thin-shell limit (which models the base of the conical shell) and the thick-shell limit (which models the tip of the conical shell), we considered a number of problems for plates, circular cylindrical shells, spherical

shells, and finally a conical shell (Figure 1). It was shown that by appropriately modifying the constitutive equations, it is possible to include enough geometrical features of the shell to predict relatively accurate results even in the thick-shell limit.

Finally, a specific problem of a conical shell of uniform thickness was considered. The heat flux on the outer surface of the shell was taken to be constant, all other surfaces of the shell were insulated, and the shell was initially at uniform temperature. Using the Cosserat equations, we were able to obtain an analytical solution for the time-dependent average temperature and temperature gradient. This solution predicted that the average temperature gradient was quite severe near the tip of the shell and developed its maximum value within a short time. Since the average temperature gradient produces a thermal moment on the shell, we expect the bending to be most severe near the shell's tip.

Future research should concentrate on determining the deformation in the conical shell produced by these thermal fields.

Appendix A

A UNIQUENESS THEOREM FOR THERMOELASTIC SHELLS WITH GENERALIZED BOUNDARY CONDITIONS

(Submitted for publication to Quarterly of Applied Mathematics)

ABSTRACT

The uniqueness of the solution of initial, mixed boundary value problems for linear thermoelastic shells is reconsidered within the context of recent developments in the thermomechanical theory of a Cosserat surface [4]. Fairly general boundary conditions are considered that allow mechanical contact with linear elastic media and thermal radiation on the boundary curve of the Cosserat surface and on the major surfaces of the shell.

1. Introduction

Motivated by the application of shell theory to contact problems, we generalize the usual boundary conditions to include mechanical contact with linear elastic media and thermal radiation. These generalized mechanical and thermal boundary conditions are applied on the boundary curve of the shell as well as on its major surfaces.* In this paper, the uniqueness proof is reconsidered within the context of these general boundary conditions.

We recall that the theory of a Cosserat surface [1] has been well established as a particularly useful model of a shell-like body that, broadly speaking, is a three-dimensional body that is "thin" in one of its dimensions. Recent developments in the theory of continuum thermodynamics [2,3] have provided the theoretical framework for developing a general thermomechanical theory of a Cosserat surface [4]. Such a theory admits finite numbers of directors $\underset{\sim}{d}_N$ $(N = 1,2...)$ and temperature field θ_N to provide limited information about the variation through the thickness of the shell of the deformation and temperature field, respectively.

For less general boundary conditions than those considered here, a uniqueness theorem has been proved for the linear isothermal theory of shells [1, Sec. 26] and for small motions superposed on a large deformation within the context of a thermoelastic theory of shells that admits a single temperature field [5]. A uniqueness theorem for a linear thermoelastic theory of shells that admits two temperature fields and two energy equations has also been proved [6], including radiation on the major surfaces, but not on the boundary curve. None of these uniqueness proofs considers mechanical contact with elastic media.

*Recall [1] that the three-dimensional boundary conditions applied to the major surfaces of the shell are incorporated into the field equations and therefore are not considered to be boundary conditions in shell theory.

Although the structure of this latter theory is different from that most recently developed for Cosserat surfaces [4], the linearized equations can be placed in a one-to-one correspondence with those of a theory that admits a single director $\underset{\sim}{d}$ and two temperature fields θ, ϕ. It therefore follows that the previous uniqueness theorem [6] applies to solutions within the context of the new linearized theory [4].

In the following sections, we state sufficient conditions to prove uniqueness of the solution of the equations of the coupled thermoelastic theory* that admits the generalized boundary conditions and that considers inhomogeneous, anisotropic elastic shells. Specifically, Section 2 records the basic equations of the linear theory of a Cosserat surface, and Section 3 discusses the generalized boundary conditions. Finally, in Section 4 we state and prove the uniqueness theorem.

2. Basic Equations

Let the material points of the Cosserat surface C be identified by means of a system of convected coordinates θ^α ($\alpha = 1,2$) and let the two-dimensional region occupied by the material surface in the present configuration at time t be denoted by c. Further, let the vector valued function $\underset{\sim}{r}$ define the position of a material point of the surface C and at each such point define the vector valued function $\underset{\sim}{d}$, called the director, and the two temperature fields θ and ϕ each referred to the present configuration. Then, a thermomechanical process of the Cosserat surface is defined by

$$\underset{\sim}{r} = \underset{\sim}{r}(\theta^\alpha,t) \quad , \quad \underset{\sim}{d} = \underset{\sim}{d}(\theta^\alpha,t) \quad , \quad [\underset{\sim}{a}_1 \; \underset{\sim}{a}_2 \; \underset{\sim}{d}] > 0 \tag{2.1a,b,c}$$

$$\theta = \theta(\theta^\alpha,t) \quad , \quad (\theta > 0) \quad , \quad \phi = \phi(\theta^\alpha,t) \quad , \tag{2.1d,e,f}$$

---

*The uniqueness theorem, with its modification for the generalized boundary conditions, is clearly applicable to the purely mechanical theory as well as the purely thermal theory.

where the tangent vectors $\mathbf{a}_\alpha$ and unit normal vector $\mathbf{a}_3$ are defined by

$$\mathbf{a}_\alpha = \frac{\partial \mathbf{r}}{\partial \theta^\alpha} \quad , \quad \mathbf{a}_\alpha \cdot \mathbf{a}_3 = 0 \quad , \quad \mathbf{a}_3 \cdot \mathbf{a}_3 = 1 \quad , \quad a^{1/2} = [\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{a}_3] > 0 \tag{2.2a,b,c,d}$$

and the condition (2.1c) ensures that the director is nowhere tangent to c. The condition (2.2d) ensures that $\mathbf{a}_i (i = 1,2,3)$ are linearly independent vectors forming a right-handed coordinate system. Thus, we may introduce a set of reciprocal vectors $\mathbf{a}^i$ such that

$$\mathbf{a}_i \cdot \mathbf{a}^j = \delta_i^j \tag{2.3}$$

where δ_i^j is the Kronecker symbol. The velocity $\mathbf{v}$, director velocity $\mathbf{w}$, and temperature gradients $\mathbf{g}$ and $\mathbf{g}_1$ may now be defined by*

$$\mathbf{v} = \dot{\mathbf{r}} \quad , \quad \mathbf{w} = \dot{\mathbf{d}} \quad , \quad \mathbf{g} = \theta_{,\alpha}\, \mathbf{a}^\alpha \quad , \quad \mathbf{g}_1 = \phi_{,\alpha}\, \mathbf{a}^\alpha \tag{2.4a,b,c,d}$$

where a superposed dot denotes time differentiation holding θ^α fixed and a comma denotes partial differentiation with respect to the coordinates θ^α. In the reference configuration, we assume that the shell is at uniform temperature Θ. Then, the reference values of the various kinematic quantities may be denoted by

$$\mathbf{r} = \mathbf{R} \quad , \quad \mathbf{d} = \mathbf{D} \quad , \quad \mathbf{a}_i = \mathbf{A}_i \quad , \quad a^{1/2} = A^{1/2} \quad , \tag{2.5a,b,c,d}$$

$$\theta = \Theta \quad , \quad \phi = 0 \quad , \tag{2.5e,f}$$

where $\mathbf{R}$, $\mathbf{D}$, $\mathbf{A}_i$ and $A^{1/2}$ depend on the coordinates θ^α only.

---

*Throughout this text, we use the usual summation convention over repeated indices. Greek indices have the range (1,2) and Latin indices have the range (1,2,3).

Let P, bounded by the closed curve ∂P, denote the region occupied by an arbitrary material portion of the surface c in the present configuration and let $\underset{\sim}{\nu}$ be the unit outward normal to ∂P. Using the notation* of [7] and referring to the present configuration, we define the following quantities: the positive mass density (mass per unit area of P) $\rho = \rho(\theta^\alpha,t)$; the contact force $\underset{\sim}{n} = \underset{\sim}{n}(\theta^\alpha,t;\ \underset{\sim}{\nu})$ and the contact director couple $\underset{\sim}{m} = \underset{\sim}{m}(\theta^\alpha,t;\ \underset{\sim}{\nu})$, each per unit length of the curve ∂P; the specific (per unit mass of P) assigned force $\underset{\sim}{f} = \underset{\sim}{f}(\theta^\alpha,t)$ and specific assigned director couple $\underset{\sim}{\ell} = \underset{\sim}{\ell}(\theta^\alpha,t)$; the intrinsic director couple $\underset{\sim}{k} = \underset{\sim}{k}(\theta^\alpha,t)$ per unit area of P; the inertia coefficients $y^1 = y^1(\theta^\alpha)$ and $y^2 = y^2(\theta^\alpha)$ which are independent of time; the specific entropies $\eta = \eta(\theta^\alpha,t)$ and $\eta_1 = \eta_1(\theta^\alpha,t)$; the specific internal rates of production of entropy $\xi = \xi(\theta^\alpha,t)$, $\xi_1 = \xi_1(\theta^\alpha,t)$ and $\bar{\xi}_1 = \bar{\xi}_1(\theta^\alpha,t)$; the entropy fluxes $k = k(\theta^\alpha,t;\ \underset{\sim}{\nu})$ and $k_1 = k_1(\theta^\alpha,t;\ \underset{\sim}{\nu})$ each per unit length acting across the curve ∂P; the specific external rates of supply of entropy $s = s(\theta^\alpha,t)$ and $s_1 = s_1(\theta^\alpha,t)$; the specific internal energy $\varepsilon = \varepsilon(\theta^\alpha,t)$; and the specific Helmholtz free energy $\psi = \psi(\theta^\alpha,t) = \varepsilon - \theta\eta - \phi\eta_1$.

For the linear theory, it is convenient to introduce the displacement vector $\underset{\sim}{u}$ and director displacement $\underset{\sim}{\delta}$ by the equations

$$\underset{\sim}{r} = \underset{\sim}{R} + \underset{\sim}{u} \quad , \quad \underset{\sim}{d} = \underset{\sim}{D} + \underset{\sim}{\delta} \quad . \tag{2.6a,b}$$

If, in the reference configuration, the shell is free of assigned fields and contact forces and director couples, then for the linear theory we assume that in the present configuration, the displacements $\underset{\sim}{u}$, $\underset{\sim}{\delta}$ and the temperatures $(\theta - \Theta)$ and ϕ are of order† $\varepsilon (0 < \varepsilon \ll 1)$ and that quadratic terms in these quantities may be neglected relative to linear

*This notation differs from that used in [1,4].

†The temporary use of this symbol for the small parameter should not be confused with the use of the same symbol elsewhere for the internal energy.

terms. With this background, we record [1] expressions for the kinematic quantities $A_{\alpha\beta}$, $B_{\alpha\beta}$, $\Lambda_{i\alpha}$, and for the strains $e_{\alpha\beta}$, γ_i, $\kappa_{i\alpha}$ in the forms

$$A_{\alpha\beta} = \underset{\sim}{A}_\alpha \cdot \underset{\sim}{A}_\beta \quad , \quad B_{\alpha\beta} = \underset{\sim}{A}_{\alpha,\beta} \cdot \underset{\sim}{A}_3 \quad , \quad \Lambda_{i\alpha} = \underset{\sim}{A}_i \cdot \underset{\sim}{D}_{,\alpha} \tag{2.7a,b,c}$$

$$e_{\alpha\beta} = \frac{1}{2}(\underset{\sim}{A}_\alpha \cdot \underset{\sim}{u}_{,\beta} + \underset{\sim}{A}_\beta \cdot \underset{\sim}{u}_{,\alpha}) \tag{2.7d}$$

$$\gamma_\alpha = \underset{\sim}{A}_\alpha \cdot \underset{\sim}{\delta} + \underset{\sim}{D} \cdot \underset{\sim}{u}_{,\alpha} \quad , \quad \gamma_3 = \underset{\sim}{A}_3 \cdot \underset{\sim}{\delta} - (\underset{\sim}{D} \cdot \underset{\sim}{A}^\alpha)(\underset{\sim}{A}_3 \cdot \underset{\sim}{u}_{,\alpha}) \quad , \tag{2.7e,f}$$

$$\kappa_{\alpha\beta} = \underset{\sim}{A}_\alpha \cdot \underset{\sim}{\delta}_{,\beta} + \underset{\sim}{D}_{,\beta} \cdot \underset{\sim}{u}_{,\alpha} \quad , \quad \kappa_{3\alpha} = \underset{\sim}{A}_3 \cdot \underset{\sim}{\delta}_{,\alpha} - (\underset{\sim}{A}^\sigma \cdot \underset{\sim}{D}_{,\alpha})(\underset{\sim}{A}_3 \cdot \underset{\sim}{u}_{,\sigma}) \tag{2.7g,h}$$

where for the linear theory all tensor quantities are referred to the base vectors $\underset{\sim}{A}_i$.

With suitable continuity assumptions, it can be shown that [1,4]

$$\underset{\sim}{n} = \underset{\sim}{N}^\alpha \nu_\alpha \quad , \quad \underset{\sim}{m} = \underset{\sim}{M}^\alpha \nu_\alpha \quad , \tag{2.8a,b}$$

$$k = \underset{\sim}{p} \cdot \underset{\sim}{\nu} = p^\alpha \nu_\alpha \quad , \quad k_1 = \underset{\sim}{p}_1 \cdot \underset{\sim}{\nu} = p_1^\alpha \nu_\alpha \quad , \tag{2.8c,d}$$

where $\nu_\alpha = \underset{\sim}{\nu} \cdot \underset{\sim}{A}_\alpha$ are the components of the normal vector $\underset{\sim}{\nu}$ and where $\underset{\sim}{N}^\alpha$, $\underset{\sim}{M}^\alpha$, $\underset{\sim}{p}$, and $\underset{\sim}{p}_1$ are independent of $\underset{\sim}{\nu}$. Further, with reference to the energy equation, the specific external rates of heat supply r and r_1; and the heat flux vectors $\underset{\sim}{q}$ and $\underset{\sim}{q}_1$ are defined by

$$r = \theta s \quad , \quad r_1 = \phi s_1 \quad , \quad \underset{\sim}{q} = \theta \underset{\sim}{p} \quad , \quad \underset{\sim}{q}_1 = \phi \underset{\sim}{p}_1 \quad . \tag{2.9a,b,c,d}$$

Now the local forms of the basic equations may be recorded as

$$\lambda = \rho\, a^{1/2} = \rho_o A^{1/2} > 0 \quad , \tag{2.10a}$$

$$\lambda(\dot{\underset{\sim}{v}} + y^1 \dot{\underset{\sim}{w}}) = \lambda \underset{\sim}{f} + (A^{1/2} \underset{\sim}{N}^\alpha)_{,\alpha} \quad , \tag{2.10b}$$

$$\lambda(y^1 \dot{\underset{\sim}{v}} + y^2 \dot{\underset{\sim}{w}}) = \lambda \underset{\sim}{\ell} - A^{1/2} \underset{\sim}{k} + (A^{1/2} \underset{\sim}{M}^{\alpha})_{,\alpha} \quad , \tag{2.10c}$$

$$\lambda \dot{\eta} = \lambda(s + \xi) - (A^{1/2} p^{\alpha})_{,\alpha} \quad , \tag{2.10d}$$

$$\lambda \dot{\eta}_1 = \lambda(s_1 + \xi_1) - (A^{1/2} p_1^{\alpha})_{,\alpha} \quad , \tag{2.10e}$$

where ρ_o is the reference value of ρ. Equation (2.10a) represents conservation of mass, (2.10b) represents the balance of linear momentum, (2.10c) represents the balance of director momentum, and (2.10d,e) represent balances of entropy. Referring the quantities $\underset{\sim}{N}^{\alpha}$, $\underset{\sim}{k}$, $\underset{\sim}{M}^{\alpha}$ to the base vectors $\underset{\sim}{A}_i$, we may write*

$$\underset{\sim}{N}^{\alpha} = N^{i\alpha} \underset{\sim}{A}_i \quad , \quad \underset{\sim}{k} = k^i \underset{\sim}{A}_i \quad , \quad \underset{\sim}{M}^{\alpha} = M^{i\alpha} \underset{\sim}{A}_i \tag{2.11a,b,c}$$

Using these definitions, the results of the balance of angular momentum become [1]

$$\acute{N}^{\beta\alpha} = \acute{N}^{\alpha\beta} = N^{\beta\alpha} - k^{\alpha} D^{\beta} - M^{\alpha\sigma} \Lambda^{\beta}_{\bullet\sigma} \quad , \tag{2.12a}$$

$$N^{3\alpha} = D^3 k^{\alpha} - D^{\alpha} k^3 + \Lambda^{3}_{\bullet\sigma} M^{\alpha\sigma} - \Lambda^{\alpha}_{\bullet\sigma} M^{3\sigma} \quad , \tag{2.12b}$$

where tensor quantities with superscripts are contravariant or mixed tensor quantities referred to $\underset{\sim}{A}_i$ and $\underset{\sim}{A}^i$.

Using appropriate constitutive equations for an elastic material and demanding that the balance of energy be identically satisfied for all processes, it can be shown [4] that

$$\acute{N}^{\alpha\beta} = \rho_o \frac{\partial\phi}{\partial e_{\alpha\beta}} \quad , \quad k^i = \rho_o \frac{\partial\phi}{\partial\gamma_i} \quad , \quad M^{i\alpha} = \rho_o \frac{\partial\phi}{\partial\kappa_{i\alpha}} \quad , \tag{2.13a,b,c}$$

$$\eta = -\frac{\partial\phi}{\partial\theta} \quad , \quad \eta_1 = \frac{\partial\phi}{\partial\theta} \quad , \tag{2.13d,e}$$

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*This notation is consistent with [7] but differs from that used in [1,4].

and that

$$\rho_o(\theta\xi + \phi\xi_1) + \underset{\sim}{p} \cdot \underset{\sim}{g} + \underset{\sim}{p}_1 \cdot \underset{\sim}{g}_1 = 0 \quad , \tag{2.14}$$

where for the linear theory, we note that ρ can be replaced by ρ_o in expressions of the type (2.13) and (2.14).

Specifically, for an inhomogeneous, anisotropic shell we assume that

$$\begin{aligned} 2\,\rho_o\,\psi = 2\,\rho_o\,\psi_1 - 2\,\beta_o\,\gamma_3(\theta - \quad) - 2\,C_4^{\alpha\beta}\,e_{\alpha\beta}(\theta - \quad) \\ - 2\,C_5^{\alpha\beta}\,\kappa_{\alpha\beta}\,\phi - \beta_3(\theta^2 - 2\,\theta \quad) - \beta_4\,\phi^2 - 2\,\beta_5\,\theta \end{aligned} \tag{2.15a}$$

$$\begin{aligned} 2\,\rho_o\,\psi_1 = C_1^{\alpha\beta\gamma\delta}\,e_{\alpha\beta}\,e_{\gamma\delta} + \alpha_4(\gamma_3)^2 + C_2^{\alpha\beta\gamma\delta}\,\kappa_{\alpha\beta}\,\kappa_{\gamma\delta} \\ + C_1^{\alpha\beta}\,\gamma_\alpha\,\gamma_\beta + C_2^{\alpha\beta}\,\kappa_{3\alpha}\,\kappa_{3\beta} + 2\,C_3^{\alpha\beta}\,e_{\alpha\beta}\,\gamma_3 \quad , \end{aligned} \tag{2.15b}$$

$$p^\alpha = -\,C_6^{\alpha\beta}\,\theta_{,\beta} \quad , \quad p_1^\alpha = -\,C_7^{\alpha\beta}\,\phi_{,\beta} \quad , \tag{2.15c,d}$$

$$\rho_o\,\theta\,\xi = C_6^{\alpha\beta}\,\theta_{,\alpha}\,\theta_{,\beta} + C_7^{\alpha\beta}\,\phi_{,\alpha}\,\phi_{,\beta} + b_2\,\phi^2 \quad , \tag{2.15e}$$

$$\rho_o\,\xi_1 = \rho_o\,\bar{\xi}_1 = -\,b_2\,\phi \quad , \tag{2.15f}$$

where the quantities

$$C_N^{\alpha\beta\gamma\delta}(N = 1,2) \quad , \quad C_N^{\alpha\beta}(N = 1,2,\ldots,7) \quad , \tag{2.16a,b}$$

$$\alpha_4 \;,\; \beta_o \;,\; \beta_3 \;,\; \beta_4 \;,\; \beta_5 \;,\; b_2 \;,\; b_3 \;, \tag{2.16c}$$

are functions of the variables

$$V = \{\theta^\alpha \;,\; A_{\alpha\beta} \;,\; B_{\alpha\beta} \;,\; D_i \;,\; \Lambda_{i\alpha} \;,\; \} \;, \tag{2.17}$$

and the tensors (2.16a,b) satisfy the symmetry restrictions

$$C_1^{\alpha\beta\gamma\delta} = C_1^{\beta\alpha\gamma\delta} = C_1^{\alpha\beta\delta\gamma} = C_1^{\gamma\delta\alpha\beta} \quad , \tag{2.18a}$$

$$C_2^{\alpha\beta\gamma\delta} = C_2^{\gamma\delta\alpha\beta} \quad , \quad C_N^{\alpha\beta} = C_N^{\beta\alpha} (N = 1\text{-}4,\ 6\text{-}7) \quad . \tag{2.18b,c}$$

The forms of the constitutive equations (2.15) are chosen to be similar to those used in the linearized theory [4], but with appropriate generalizations to allow for anisotropic, inhomogeneous response and to satisfy the reduced energy equation (2.14) without approximation. Specifically, equations (2.15c-f) satisfy the nonlinear form of the reduced energy equation (2.14), with ρ_o replaced by ρ in each of the expressions. In particular, we note from (2.15e) that ξ is of order ε^2 and therefore may be neglected in equation (2.10d). Furthermore, we note that the constitutive equations (2.15) automatically satisfy restrictions for modeling a shell that has symmetric response about its reference surface [1, Sec. 13 γ]. Now, with the help of the restrictions (2.13), (2.18) and the constitutive equations (2.15), we conclude that

$$\acute{N}^{\alpha\beta} = C_1^{\alpha\beta\gamma\delta}\, e_{\gamma\delta} + C_3^{\alpha\beta}\, \gamma_3 - C_4^{\alpha\beta}(\theta - \Theta) \quad , \tag{2.19a}$$

$$k^{\alpha} = C_1^{\alpha\beta}\, \gamma_\beta \quad , \quad k^3 = \alpha_4\, \gamma_3 + C_3^{\alpha\beta}\, e_{\alpha\beta} - \beta_o(\theta - \Theta) \quad , \tag{2.19b,c}$$

$$M^{\alpha\beta} = C_2^{\alpha\beta\gamma\delta}\, \kappa_{\gamma\delta} - C_5^{\alpha\beta}\, \phi \quad , \quad M^{3\alpha} = C_2^{\alpha\beta}\, \kappa_{3\beta} \quad , \tag{2.19d,e}$$

$$\rho_o \eta = \beta_o\, \gamma_3 + C_4^{\alpha\beta}\, e_{\alpha\beta} + \beta_3(\theta - \Theta) + \beta_5 \quad , \quad \rho_o \eta_1 = C_5^{\alpha\beta}\, \kappa_{\alpha\beta} + \beta_4\, \phi \quad , \tag{2.19f,g}$$

$$2\, \rho_o \varepsilon = 2\, \rho_o \phi_1 + 2 \quad \beta_o\, \gamma_3 + 2 \quad C_4^{\alpha\beta}\, e_{\alpha\beta} + \beta_3\, \theta^2 + \beta_4\, \phi^2 \tag{2.19h}$$

Finally, we recall [4] that for such an elastic shell the only nontrivial statements of the second law of thermodynamics take the forms

$$\rho_o \, \phi \, \bar{\xi}_1 + \underset{\sim}{p} \cdot \underset{\sim}{g} + \underset{\sim}{p}_1 \cdot \underset{\sim}{g}_1 \leq 0 \quad , \tag{2.20a}$$

$$\theta(t) - \theta_1 > 0 \quad \text{whenever} \quad \varepsilon(t) - \varepsilon_1 > 0 \quad . \tag{2.20b}$$

The statement (2.20a) corresponds to the classical heat conduction inequality and is assumed valid for all equilibrium displacement and temperature fields. Further, the statement (2.20b) is assumed valid when the Cosserat surface is at rest and the three-dimensional temperature field is spatially uniform so that $\theta = \theta(t)$, $\phi = 0$. In (2.20b), ε_1 and θ_1 correspond to the internal energy and uniform temperature of the shell during some period of time up to t_1 when the shell has been at rest and in thermal equilibrium.

3. Boundary and Initial Conditions

This theory, which is developed by direct approach, may be brought into a one-to-one correspondence with the three-dimensional theory by assuming that the position vector $\underset{\sim}{p}^*$ of a point in the shell and the temperature field θ^* admit the representations.

$$\underset{\sim}{p}^* = \underset{\sim}{p}^*(\theta^\alpha, \theta^3, t) = \underset{\sim}{r}(\theta^\alpha, t) + \theta^3 \, \underset{\sim}{d}(\theta^\alpha, t) \quad , \tag{3.1a}$$

$$\theta^* = \theta^*(\theta^\alpha, \theta^3, t) = \theta(\theta^\alpha, t) + \theta^3 \, \phi(\theta^\alpha, t) \quad , \tag{3.1b}$$

where θ^3 is a coordinate through the thickness of the shell. Without loss in generality, we may define the top surface ∂P^+ of the shell by $\theta^3 = h/2$, where h is a constant having the dimensions of length; then we may write the displacement $\underset{\sim}{u}^+$ and temperature difference $(\theta^+ - \Theta)$ on ∂P^+ by

$$\underset{\sim}{u}^+ = \underset{\sim}{u} + \frac{h}{2} \underset{\sim}{\delta} \quad , \quad (\theta^+ - \Theta) = (\theta - \Theta) + \frac{h}{2} \phi \quad . \tag{3.2a,b}$$

Similarly, we may define the bottom surface ∂P^- of the shell by $\theta^3 = -h/2$ and write the displacement $\underset{\sim}{u}^-$ and temperature difference $(\theta^- - \Theta)$ on ∂P^- by

$$\underset{\sim}{u}^- = \underset{\sim}{u} - \frac{h}{2}\underset{\sim}{\delta} \quad , \quad (\theta^- - \circledR) = (\theta - \circledR) - \frac{h}{2}\phi \quad . \qquad (3.3a,b)$$

In this regard, we note that the specification that the major surfaces of the shell are defined by $\theta^3 = \pm h/2$, where h is constant, does not restrict the theory to that associated with a shell of uniform thickness; this is because the director $\underset{\sim}{D}$ in the reference configuration may be specified as a function of θ^α.

With the help of (2.12) and the constitutive equations (2.15c,d,f), (2.19a-g) and the strain-displacement relations (2.7), equations (2.10), with $\xi = 0$, represent a system of equations to determine the unknowns $\underset{\sim}{u}$, $\underset{\sim}{\delta}$, θ, ϕ. These equations must be solved subject to certain initial and boundary conditions. Here, we define the initial conditions at each point of the region P by specifying

$$\underset{\sim}{u} = \underset{\sim}{u}_o(\theta^\alpha) \quad , \quad \underset{\sim}{\delta} = \underset{\sim}{\delta}_o(\theta^\alpha) \quad , \quad \theta = \theta_o(\theta^\alpha) \quad , \quad \phi = \phi_o(\theta^\alpha) \quad , \qquad (3.4a,b,c,d)$$

$$\dot{\underset{\sim}{u}} = \underset{\sim}{v}_o(\theta^\alpha) \quad , \quad \dot{\underset{\sim}{\delta}} = \underset{\sim}{w}_o(\theta^\alpha) \quad , \quad \text{at } t = t_o \quad . \qquad (3.4e,f)$$

where $\underset{\sim}{u}_o$, $\underset{\sim}{v}_o$, $\underset{\sim}{\delta}_o$, $\underset{\sim}{w}_o$, θ_o, ϕ_o are specified functions of θ^α only. Mixed* boundary conditions at each point** s of the boundary ∂P may be defined by specifying either:

$$\underset{\sim}{n} = \bar{\underset{\sim}{n}}(s,t) \quad \text{or} \quad \underset{\sim}{u} = \bar{\underset{\sim}{u}}(s,t) \quad , \qquad (3.5a,b)$$

$$\underset{\sim}{m} = \bar{\underset{\sim}{m}}(s,t) \quad \text{or} \quad \underset{\sim}{\delta} = \bar{\underset{\sim}{\delta}}(s,t) \quad , \qquad (3.5c,d)$$

$$k = \bar{k}(s,t) \quad \text{or} \quad \theta = \bar{\theta}(s,t) \quad , \qquad (3.5e,f)$$

*Mixed-mixed boundary conditions may be specified but are not considered explicitly.

**The temporary use of the symbol s for a point on ∂P should not be confused with the use of the same symbol elsewhere for the external entropy supply.

$$k_1 = \bar{k}_1(s,t) \quad \text{or} \quad \phi = \bar{\phi}(s,t) \quad , \quad \text{for } t \, \varepsilon [t_o, \infty) \tag{3.5g,h}$$

More general boundary conditions than (3.5), which include mechanical contact with a linear elastic media or thermal radiation, may be written at each point s of ∂P in the generalized form*

$$\underset{\sim}{n} + \underset{\sim}{B}(s)\, \underset{\sim}{u} + \underset{\sim}{C}(s,t) = 0 \quad , \quad \underset{\sim}{m} + \underset{\sim}{B}_1(s)\, \underset{\sim}{\delta} + \underset{\sim}{C}_1(s,t) = 0 \quad , \tag{3.6a,b}$$

$$k - B(s,t)(\theta - \Theta) + C(s,t) = 0 \quad , \quad k_1 - B_1(s,t)\, \phi + C_1(s,t) = 0 \quad , \tag{3.6c,d}$$

where $\underset{\sim}{B}$ and $\underset{\sim}{B}_1$ are assumed to be continuous symmetric three-dimensional tensor functions of s and are independent of time; B and B_1 are continuous scalar functions of (s,t); $\underset{\sim}{C}$ and $\underset{\sim}{C}_1$ are continuous vector functions of (s,t); and C and C_1 are continuous scalar functions of (s,t). For later convenience, we define the scalars

$$J_1 = \frac{1}{2}\, \underset{\sim}{u} \cdot \underset{\sim}{B}\, \underset{\sim}{u} \quad , \quad J_2 = \frac{1}{2}\, \underset{\sim}{\delta} \cdot \underset{\sim}{B}_1\, \underset{\sim}{\delta} \quad . \tag{3.7a,b}$$

A boundary condition of the type (3.5a) can be obtained trivially from (3.6a) by setting $\underset{\sim}{B} = 0$, and $\underset{\sim}{C} = -\bar{\underset{\sim}{n}}(s,t)$, and a boundary condition of the type (3.5b) can be obtained from (3.6b) by setting the tensor $\underset{\sim}{B}$ equal to a scalar b times the identity tensor $\underset{\sim}{I}(\underset{\sim}{B} = b\, \underset{\sim}{I})$ and $\underset{\sim}{C} = -b\, \bar{\underset{\sim}{u}}(s,t)$, and then taking the limit as b approaches infinity.

Recall from [4] that the assigned fields $\underset{\sim}{f}$, $\underset{\sim}{\ell}$, s, and s_1 include contributions from both the effects of three-dimensional body force and external entropy supply as well as from the effects of surface tractions and entropy flux on the major surfaces of the shell. In view of the specification $\theta^3 = \pm\, h/2$ defining the major surfaces, we may write these assigned fields in the forms

*A condition of the type (3.6c) has previously been considered for the three-dimensional theory [8, Sec. 5.7].

$$\lambda \underset{\sim}{f} = \lambda \bar{\underset{\sim}{f}} + A^{1/2}(b^{+} \underset{\sim}{t}^{+} + b^{-} \underset{\sim}{t}^{-}) \quad , \tag{3.8a}$$

$$\lambda \underset{\sim}{\ell} = \lambda \bar{\underset{\sim}{\ell}} + A^{1/2}(\frac{h}{2})(b^{+} \underset{\sim}{t}^{+} - b^{-} \underset{\sim}{t}^{-}) \quad , \tag{3.8b}$$

$$\lambda s = \lambda \bar{s} - A^{1/2}(b^{+} k^{+} + b^{-} k^{-}) \quad , \tag{3.8c}$$

$$\lambda s_1 = \lambda \bar{s}_1 - A^{1/2}(\frac{h}{2})(b^{+} k^{+} - b^{-} k^{-}) \quad , \tag{3.8d}$$

where $\bar{\underset{\sim}{f}}$, $\bar{\underset{\sim}{\ell}}$, $\bar{\underset{\sim}{s}}$, $\bar{s}_1$ are considered to be specified assigned fields associated with the three-dimensional body force and external supply of entropy, b^+ and b^- are positive scalar functions of the variables (2.17) and are independent of time, $\underset{\sim}{t}^+$ is the traction vector and k^+ is the entropy flux on the major surface ∂P^+, and $\underset{\sim}{t}^-$ is the traction vector and k^- is the entropy flux on the major surface ∂P^-. To allow mechanical contact of the major surfaces with an elastic media and to allow thermal radiation, we assume that at each point of* ∂P^+

$$\underset{\sim}{t}^{+} + \underset{\sim}{B}^{+} \underset{\sim}{u}^{+} + \underset{\sim}{C}^{+} = 0 \quad , \quad k^{+} - B^{+}(\theta^{+} - \Theta) + C^{+} = 0 \tag{3.9a,b}$$

and at each point of ∂P^-

$$\underset{\sim}{t}^{-} + \underset{\sim}{B}^{-} \underset{\sim}{u}^{-} + \underset{\sim}{C}^{-} = 0 \quad , \quad k^{-} - B^{-}(\theta^{-} - \Theta) + C^{-} = 0 \quad , \tag{3.10a,b}$$

where $\underset{\sim}{B}^+$ and $\underset{\sim}{B}^-$ are continuous, symmetric, three-dimensional second order tensor functions of the variables V in (2.17) and are independent of time; $\underset{\sim}{C}^+$ and $\underset{\sim}{C}^-$ are continuous vector functions of the variables (V,t); and B^+, B^-, C^+, C^- are continuous scalar functions of the variables (V,t). For later convenience, we define the scalars

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*Note that the range of the convected coordinates θ^α on the major surfaces ∂P^+ and ∂P^- is the same as that on the reference surface P of the Cosserat surface.

$$J_3 = \frac{1}{2}\underset{\sim}{u}^+ \cdot \underset{\sim}{B}^+ \underset{\sim}{u}^+ \quad , \quad J_4 = \frac{1}{2}\underset{\sim}{u}^- \cdot \underset{\sim}{B}^- \underset{\sim}{u}^- \quad . \tag{3.11a,b}$$

Depending on the nature of the problem to be considered, the quantities $\underset{\sim}{t}^+$, $\underset{\sim}{t}^-$, k^+, k^-, $\underset{\sim}{u}^+$, $\underset{\sim}{u}^-$, θ^+, θ^- are either specified or to be determined by the field equations. For example, if we were to consider a slightly more general boundary condition of the type (3.9a), which specified the tangential components of $\underset{\sim}{t}^+$ and the normal component of $\underset{\sim}{u}^+$, then the normal component of $\underset{\sim}{t}^+$ and the tangential components of $\underset{\sim}{u}^+$ would be determined by the field equations. This is similar to the use of such quantities in the theory of laminated composite plates [9].

4. A Uniqueness Theorem

We now state the following uniqueness theorem: Let $\underset{\sim}{u}$, $\underset{\sim}{\delta}$, θ, ϕ be displacements and temperature fields that satisfy the above-mentioned linear field equations, constitutive equations, and statement of the second law of thermodynamics on $P \times [t_o, \infty)$, and satisfy the initial conditions on P at $t = t_o$, the boundary conditions on $\partial P \times [t_o, \infty)$, conditions of the type (3.9) on $\partial P^+ \times [t_o, \infty)$ and (3.10) on $\partial P^- \times [t_o, \infty)$, for prescribed values of the assigned force $\bar{\underset{\sim}{f}}$, director couple $\bar{\underset{\sim}{\ell}}$, and external supplies of entropy $\bar{s}$ and $\bar{s}_1$. Then, provided the specific kinetic energy K, defined by

$$K = K(\underset{\sim}{v},\underset{\sim}{w}) = \frac{1}{2}(\underset{\sim}{v} \cdot \underset{\sim}{v} + 2\, y^1\, \underset{\sim}{v} \cdot \underset{\sim}{w} + y^2\, \underset{\sim}{w} \cdot \underset{\sim}{w}) \quad , \tag{4.1}$$

is positive definite, the specific heats β_3 and β_4 in (2.15a) are positive scalars, the portion of the Helmholtz free energy ψ_1 in (2.15b) is positive semi-definite, and the scalars J_1, J_2, J_3, J_4, B, B_1, B^+, B^- in (3.6), (3.7), (3.9)-(3.11) are positive semi-definite, there exists at most one set of functions $\underset{\sim}{u}$, $\underset{\sim}{\delta}$, θ, ϕ that satisfies the strain-displacement relations (2.8), the field equations (2.10) (with $\xi = 0$ in 2.10d) and (2.12), the constitutive equations (2.15) and (2.19), the restriction (2.20a), initial conditions (3.4), boundary conditions (3.6), and conditions (3.9) and (3.10), are of class C^1 on $\partial P \times [t_o, \infty)$, and are of class C^2 on $P \times [t_o, \infty)$. For convenience, the restrictions

stated above may be written in the mathematical forms*

$$y^2 - (y^1)^2 > 0 \quad , \quad \beta_3 > 0 \quad , \quad \beta_4 > 0 \quad , \quad \phi_1 \geq 0 \quad , \qquad (4.2a,b,c,d)$$

$$J_1 \geq 0 \quad , \quad J_2 \geq 0 \quad , \quad J_3 \geq 0 \quad ; \quad J_4 \geq 0 \quad , \qquad (4.2e,f,g,h)$$

$$B \geq 0 \quad , \quad B_1 \geq 0 \quad , \quad B^+ \geq 0 \quad , \quad B^- \geq 0 \quad . \qquad (4.2i,j,k,\ell)$$

Apart from the discussion of the generalized boundary conditions, our method of proof is nearly identical to that used in Reference [6]. Specifically, we assume the existence of two different solutions of the above-stated initial, mixed boundary-value problem, form the difference solution, and use a consequence of the field equations to prove that the difference solution is the null solution. Let us denote the two solutions by the sets of quantities

$$U = \{\underset{\sim}{u}, \underset{\sim}{\delta}, \theta, \phi, \underset{\sim}{n}, \underset{\sim}{k}, \underset{\sim}{m}, \eta, \eta_1, \xi_1, \underset{\sim}{p}, \underset{\sim}{p}_1, \underset{\sim}{t}^+, \underset{\sim}{t}^-, k^+, k^-\} \qquad (4.3a)$$

$$U' = \{\underset{\sim}{u}', \underset{\sim}{\delta}', \theta', \phi', \underset{\sim}{n}', \underset{\sim}{k}', \underset{\sim}{m}', \eta', \eta_1', \xi_1', \underset{\sim}{p}', \underset{\sim}{p}_1', \underset{\sim}{t}'^+, \underset{\sim}{t}'^-, k'^+, k'^-\} \qquad (4.3b)$$

and form the difference solution

$$\hat{U} = U - U' \quad . \qquad (4.4)$$

It follows that the difference solution satisfies the following: the field equations

$$\lambda(\ddot{\hat{\underset{\sim}{v}}} + y^1 \ddot{\hat{\underset{\sim}{w}}}) = A^{1/2}(b^+ \hat{\underset{\sim}{t}}^+ + b^- \hat{\underset{\sim}{t}}^-) + (A^{1/2} \hat{\underset{\sim}{N}}^\alpha)_{,\alpha} \quad , \qquad (4.5a)$$

---

*The restriction (4.2b) is consistent with the condition (2.20b).

$$\lambda(y^1 \hat{\mathring{\underset{\sim}{v}}} + y^2 \hat{\mathring{\underset{\sim}{w}}}) = A^{1/2}(\tfrac{h}{2})(b^+ \hat{\underset{\sim}{t}}^+ - b^- \hat{\underset{\sim}{t}}^-) - A^{1/2} \hat{\underset{\sim}{k}} + (A^{1/2} \hat{\underset{\sim}{M}}^\alpha)_{,\alpha} \tag{4.5b}$$

$$\lambda\hat{\mathring{\eta}} = -A^{1/2}(b^+ \hat{k}^+ + b^- \hat{k}^-) - (A^{1/2} \hat{p}^\alpha)_{,\alpha} \quad , \tag{4.5c}$$

$$\lambda\hat{\mathring{\eta}}_1 = \lambda\hat{\xi}_1 - A^{1/2}(\tfrac{h}{2})(b^+ \hat{k}^+ - b^- \hat{k}^-) - (A^{1/2} \hat{p}_1^\alpha)_{,\alpha} \quad , \tag{4.5d}$$

on $P \times [t_o, \infty)$, the restriction

$$\rho_o \hat{\phi} \hat{\xi} + \hat{\underset{\sim}{p}} \cdot \hat{\underset{\sim}{g}} + \hat{\underset{\sim}{p}}_1 \cdot \hat{\underset{\sim}{g}}_1 \leq 0 \tag{4.6}$$

on $P \times [t_o, \infty)$, the initial conditions

$$\hat{\underset{\sim}{u}} = 0 \quad , \quad \hat{\underset{\sim}{\delta}} = 0 \quad , \quad \hat{\theta} = 0 \quad , \quad \hat{\phi} = 0 \quad , \quad \hat{\mathring{\underset{\sim}{u}}} = 0 \quad , \quad \hat{\mathring{\underset{\sim}{\delta}}} = 0 \tag{4.7a-f}$$

on P at $t = t_o$, the boundary conditions

$$\hat{\underset{\sim}{n}} + \underset{\sim}{B} \hat{\underset{\sim}{u}} = 0 \quad , \quad \hat{\underset{\sim}{m}} + \underset{\sim}{B}_1 \hat{\underset{\sim}{\delta}} = 0 \quad , \tag{4.8a,b}$$

$$\hat{k} - B \hat{\theta} = 0 \quad , \quad \hat{k}_1 - B_1 \hat{\phi} = 0 \quad , \tag{4.8c,d}$$

on $\partial P \times [t_o, \infty)$, the conditions

$$\hat{\underset{\sim}{t}}^+ + \underset{\sim}{B}^+ \hat{\underset{\sim}{u}}^+ = 0 \quad , \quad \hat{k}^+ - B^+ \hat{\theta}^+ = 0 \quad , \tag{4.9a,b}$$

on $\partial P^+ \times [t_o, \infty)$, and the conditions

$$\hat{\underset{\sim}{t}}^- + \underset{\sim}{B}^- \hat{\underset{\sim}{u}}^- = 0 \quad , \quad \hat{k}^- - B^- \hat{\theta}^- = 0 \quad , \tag{4.10a,b}$$

on $\partial P^- \times [t_o, \infty)$.

Multiplying (4.5c) by $\hat{\theta}$, (4.5d) by $\hat{\phi}$, adding the results together, integrating over the region P, using the divergence theorem and the conditions (4.8c,d), (4.9b), and (4.10b), we obtain

$$\int_P \rho_o(\hat{\theta}\,\dot{\hat{\eta}} + \hat{\phi}\,\dot{\hat{\eta}}_1)d\sigma = \int_P(\rho_o\,\hat{\phi}\,\hat{\xi}_1 + \hat{\underset{\sim}{p}}\cdot\hat{\underset{\sim}{g}} + \hat{\underset{\sim}{p}}_1\cdot\hat{\underset{\sim}{g}}_1)d\sigma$$

$$- \int_P[b^+\,B^+(\hat{\theta}^+)^2 + b^-\,B^-\,(\hat{\theta}^-)^2]d\sigma$$

$$- \int_{\partial P}(B\,\hat{\theta}^2 + B_1\,\hat{\phi}^2)ds \quad , \qquad (4.11)$$

where $d\sigma$ is the area element on P and ds is the arc length on ∂P. Taking the inner product of (4.5a) with $\hat{\underset{\sim}{v}}$, (4.5b) with $\hat{\underset{\sim}{w}}$, adding the results together, integrating over the region P, using the divergence theorem and the conditions (4.8a,b), (4.9a), and (4.10a), we deduce the expression

$$\dot{E} = \int_P \rho_o(\hat{\theta}\,\dot{\hat{\eta}} + \hat{\phi}\,\dot{\hat{\eta}}_1)d\sigma \qquad (4.12)$$

where

$$E = \int_P [\rho_o(\hat{\phi}_1 + \hat{K}) + \frac{1}{2}\beta_3\,\hat{\theta}^2 + \frac{1}{2}\beta_4\,\hat{\phi}^2 + b^+\,\hat{J}_3 + b^-\,\hat{J}_4]d\sigma$$

$$+ \int_{\partial P}(\hat{J}_1 + \hat{J}_2)ds \quad , \qquad (4.13a)$$

$$\hat{\phi}_1 = \phi_1(\hat{e}_{\alpha\beta}, \hat{\gamma}_i, \hat{\kappa}_{i\alpha}) \quad , \quad \hat{K} = K(\hat{\underset{\sim}{v}}, \hat{\underset{\sim}{w}}) \quad , \quad \hat{J}_1 = J_1(\hat{\underset{\sim}{u}}) \quad , \qquad (4.13b\text{–}d)$$

$$\hat{J}_2 = J_2(\hat{\underset{\sim}{\delta}}) \quad , \quad \hat{J}_3 = J_3(\hat{u}^+) \quad , \quad \hat{J}_4 = J_4(\hat{u}^-) \qquad (4.13e,f,g)$$

and where the functions ϕ_1, K, $J_1 - J_4$, are defined by (2.15b), (4.1), (3.7), and (3.11), respectively. Now from (4.11)-(4.13) and the restrictions (4.2), we realize that $E \geq 0$ and $\dot{E} \leq 0$. Using the initial conditions (4.7), we obtain the result that E = 0 for all time $t\,\epsilon[t_o, \infty)$ and therefore

$$\hat{\underset{\sim}{v}} = 0 \quad , \quad \hat{\underset{\sim}{w}} = 0 \quad , \quad \hat{\theta} = 0 \quad , \quad \hat{\phi} = 0 \quad . \qquad (4.14a,b,c,d)$$

Integrating (4.14a,b) and using the initial condition (4.7), we have

$$\hat{\underset{\sim}{u}} = 0 \quad , \quad \hat{\underset{\sim}{\delta}} = 0 \quad , \qquad (4.15a,b)$$

Finally, substituting (4.14c,d) and (4.15a,b) into the conditions (4.9), (4.10), we conclude that

$$\hat{\underset{\sim}{t}}^{+} = 0 \quad , \quad \hat{\underset{\sim}{t}}^{-} = 0 \quad , \quad \hat{k}^{+} = 0 \quad , \quad \hat{k}^{-} = 0 \tag{4.16a,b,c,d}$$

which completes the proof.

To prove uniqueness for thermoelastic statics, we need slightly stronger conditions than (2.20a) and (4.2d). Specifically, we retain the restrictions (4.2e-ℓ) and assume that ψ_1 is positive definite and that the expression on the left-hand side of (2.20a) is negative definite so that

$$\psi_1 > 0 \quad , \quad \rho_o \, \phi \, \bar{\xi}_1 + \underset{\sim}{p} \cdot \underset{\sim}{g} + \underset{\sim}{p}_1 \cdot \underset{\sim}{g}_1 < 0 \tag{4.17a,b}$$

where ψ_1 in (4.17a) vanishes only when the mechanical fields $e_{\alpha\beta}$, γ_i, $\kappa_{i\alpha}$ vanish and (4.17b) vanishes only when the thermal fields $\theta_{,\alpha}$, ϕ, $\phi_{,\alpha}$ vanish. Furthermore, we require the temperature θ to be specified on at least one point on the boundary of the shell. This can be done by specifying θ on ∂P, θ^{+} on ∂P^{+}, or θ^{-} on ∂P^{-}.

Now, for thermoelastic statics the thermal equations (4.5c,d) are uncoupled from the mechanical equations (4.5a,b) and the expression (4.11) can be derived with the left hand side vanishing. It follows that*

$$\rho_o \, \hat{\phi} \, \hat{\xi}_1 + \hat{\underset{\sim}{p}} \cdot \hat{\underset{\sim}{g}} + \hat{\underset{\sim}{p}}_1 \cdot \hat{\underset{\sim}{g}}_1 = 0 \quad , \tag{4.18}$$

from which we conclude that

$$\hat{\underset{\sim}{g}} = 0 \quad , \quad \hat{\phi} = 0 \quad . \tag{4.19a,b}$$

*Recall from (2.15f) that $\bar{\xi}_1 = \xi_1$.

Integrating (4.19a) and using the specification of θ at a point of the boundary of the shell, we have

$$\hat{\theta} = 0 \quad . \tag{4.20}$$

Substituting (4.19b) and (4.20) into the conditions (4.9b) and (4.10b), we deduce that

$$\hat{k}^{+} = 0 \quad , \quad \hat{k}^{-} = 0 \quad , \tag{4.21a,b}$$

which completes the proof for the thermal fields. In view of the results (4.19b) and (4.20), we may take the inner product of the equilibrium form of equations (4.5a) and (4.5b) with $\hat{\underset{\sim}{u}}$ and $\hat{\underset{\sim}{\delta}}$, respectively, and derive the expression

$$\int_{P} (\rho_{o} \hat{\phi}_{1} + b^{+} \hat{J}_{3} + b^{-} \hat{J}_{4}) d\sigma + \int_{\partial P} (\hat{J}_{1} + \hat{J}_{2}) ds = 0 \quad . \tag{4.22}$$

It follows that $\hat{\phi}_{1} = 0$ so that

$$\hat{e}_{\alpha\beta} = 0 \quad , \quad \hat{\gamma}_{i} = 0 \quad , \quad \hat{\kappa}_{i\alpha} = 0 \quad . \tag{4.23a,b,c}$$

Hence, the displacements are unique to within a linear superposed rigid body displacement. If this arbitrariness is removed, then the displacements will be unique

$$\hat{\underset{\sim}{u}} = 0 \quad , \quad \hat{\underset{\sim}{\delta}} = 0 \tag{4.24a,b}$$

and from the conditions (4.9a) and (4.10a), we can conclude that

$$\hat{\underset{\sim}{t}}^{+} = 0 \quad , \quad \hat{\underset{\sim}{t}}^{-} = 0 \quad , \tag{4.25a,b}$$

which completes the proof for the mechanical fields.

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Appendix B

A NONLINEAR CONSTRAINED THEORY OF SHELLS THAT INCLUDES TANGENTIAL SHEAR DEFORMATION

(Submitted for publication to ASME Journal of Applied Mechanics)

Introduction

Within the context of theories of deformable bodies, a constrained material is a material that can experience only a restricted class of motions. Various constrained theories of shells have been developed mainly because the system of equations characterizing a constrained theory is simpler than that characterizing the general theory. For shells, it is common to develop constrained theories that exclude one or both of the following two types of deformations: (a) normal extension and (b) tangential shear deformation. The terminology normal extension and tangential shear deformation is used instead of the usual terminology "transverse normal strain," and "transverse shear deformation" because it is more descriptive when considering nonlinear deformation of a shell as opposed to linear deformation of a plate.

To discuss constrained theories of shells of the type considered here, it is particularly convenient to model the shell as a Cosserat surface [1]. Although various restricted or constrained theories of shells [1, Sec. 10; 2,3,4] have been developed, we choose to focus attention on three constrained theories. For constrained theory I, we exclude both deformations (a) and (b) so that a material fiber that is initially normal to the undeformed reference surface of the shell remains normal to the deformed reference surface and has constant length. A nonlinear theory of this type has been developed as a restricted theory [1, Sec. 10], and it has been observed that it reduces to the classical linear Kirchhoff-Love plate theory. For constrained theory II, we exclude only deformation (b) so that a material fiber that is initially normal to the undeformed reference surface of the shell remains normal to the deformed reference surface, but is allowed to extend or contract in length. A nonlinear theory of this type has also been developed [2]. For constrained theory III, we exclude only deformation (a) so that a material fiber that is initially normal to the undeformed reference surface of the shell is allowed to deform away from the normal to the deformed reference surface, but the component of the material fiber normal to the deformed reference surface remains constant.

Although a theory of this type can be reduced to a linearized theory of a plate that includes "transverse" shear deformation [5], a nonlinear version of this theory does not appear to have been previously developed.

Our purpose here is to develop a nonlinear version of constrained theory III. Because the constraint associated with this theory is mechanical in nature, we will confine attention to the purely mechanical theory.* In the following sections, we record the basic equations and discuss the constraint and constraint response associated with theory III. Next, the boundary conditions are discussed, with a few comments on the linear theory, and finally the initial conditions are stated.

Basic Equations

In this section, we use the notation in [2] and briefly record the basic equations appropriate for a constrained version of the theory of a Cosserat surface. For a complete discussion of this theory, we refer the reader to [1,6]. Let the material points of the Cosserat surface C be identified by means of a system of convected coordinates $\theta^\alpha(\alpha = 1,2)$ and let the two-dimensional region occupied by the material surface in the present configuration at time t be denoted by c. Further, with reference to the present configuration, let the vector valued function $\underset{\sim}{r}$ define the position of a material point of C and at each such point define the vector valued function $\underset{\sim}{d}$, called the director. Then, a motion of the Cosserat surface is defined by

$$\underset{\sim}{r} = \underset{\sim}{r}(\theta^\alpha,t) \quad , \quad \underset{\sim}{d} = \underset{\sim}{d}(\theta^\alpha,t) \quad , \quad [\underset{\sim}{a}_1 \ \underset{\sim}{a}_2 \ \underset{\sim}{d}] > 0 \qquad \text{(1a,b,c)}$$

where the tangent vectors $\underset{\sim}{a}_\alpha$ and the unit normal vector $\underset{\sim}{a}_3$ are defined by

*A thermomechanical theory of the type [6] could be developed by appropriately modifying the constitutive equations.

$$\mathbf{a}_\alpha = \frac{\partial \mathbf{r}}{\partial \theta^\alpha} \quad , \quad \mathbf{a}_\alpha \cdot \mathbf{a}_3 = 0 \quad , \quad \mathbf{a}_3 \cdot \mathbf{a}_3 = 1 \quad , \quad a^{1/2} = [\mathbf{a}_1 \; \mathbf{a}_2 \; \mathbf{a}_3] > 0 \quad . \tag{2a,b,c,d}$$

The velocity $\mathbf{v}$ and director velocity $\mathbf{w}$ are defined by

$$\mathbf{v} = \dot{\mathbf{r}} \quad , \quad \mathbf{w} = \dot{\mathbf{d}} \quad , \tag{3a,b}$$

where a superposed dot denotes material differentiation holding θ^α fixed.

In the reference configuration, the various kinematic quantities may be denoted by*

$$\mathbf{r} = \mathbf{R} \quad , \quad \mathbf{d} = \mathbf{D} = D\,\mathbf{A}_3 \quad , \quad \mathbf{a}_i = \mathbf{A}_i \quad , \quad a^{1/2} = A^{1/2} \quad , \tag{4a,b,c,d}$$

where $\mathbf{R}$, $\mathbf{D}$, $\mathbf{A}_i$, $A^{1/2}$ depend on the coordinates θ^α only. The specification (4b) is made without loss in generality. For later convenience, we recall [1] the kinematic definitions

$$a_{\alpha\beta} = \mathbf{a}_\alpha \cdot \mathbf{a}_\beta \quad , \quad d_i = \mathbf{a}_i \cdot \mathbf{d} \quad , \quad \lambda_{i\alpha} = \mathbf{a}_i \cdot \mathbf{d}_{,\alpha} \tag{5a,b,c}$$

$$e_{\alpha\beta} = \frac{1}{2}(a_{\alpha\beta} - A_{\alpha\beta}) \quad , \quad \gamma_i = d_i - D_i \quad , \quad \kappa_{i\alpha} = \lambda_{i\alpha} - \Lambda_{i\alpha} \tag{5d,e,f}$$

$$b_{\alpha\beta} = \mathbf{a}_{\alpha,\beta} \cdot \mathbf{a}_3 \quad , \tag{5g}$$

where $A_{\alpha\beta}$, D_i, $\Lambda_{i\alpha}$, $B_{\alpha\beta}$ are the reference values of $a_{\alpha\beta}$, d_i, $\lambda_{i\alpha}$, $b_{\alpha\beta}$, respectively; $e_{\alpha\beta}$, γ_i, $\kappa_{i\alpha}$ are strains measured relative to the reference configuration; and a comma denotes partial differentiation with respect to θ^α.

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*Throughout this text, Greek indices have the range (1,2) and Latin indices have the range (1,2,3).

Let $\mathcal{P}$, bounded by the closed curve $\partial\mathcal{P}$, denote the region occupied by an arbitrary material portion of the surface c in the present configuration and let $\underset{\sim}{\nu}$ be the unit outward normal to $\partial\mathcal{P}$. Using the notation* of [2] and referring to the present configuration, we define the following quantities: the mass density (mass per unit area of $\mathcal{P}$) $\rho = \rho(\theta^\alpha,t)$; the contact force $\underset{\sim}{n} = \underset{\sim}{n}(\theta^\alpha,t;\ \underset{\sim}{\nu})$ and the contact director couple $\underset{\sim}{m} = \underset{\sim}{m}(\theta^\alpha,t;\ \underset{\sim}{\nu})$, each per unit length of the curve $\partial\mathcal{P}$; the specific (per unit mass of $\mathcal{P}$) assigned force $\underset{\sim}{f} = \underset{\sim}{f}(\theta^\alpha,t)$ and specific assigned director couple $\underset{\sim}{\ell} = \underset{\sim}{\ell}(\theta^\alpha,t)$; the intrinsic director couple $\underset{\sim}{k} = \underset{\sim}{k}(\theta^\alpha,t)$ per unit area of $\mathcal{P}$; and the inertia coefficients $y^1 = y^1(\theta^\alpha)$ and $y^2 = y^2(\theta^\alpha)$, which are independent of time. With suitable continuity assumptions, it can be shown [1] that

$$\underset{\sim}{n} = \underset{\sim}{N}^\alpha \nu_\alpha \quad , \quad \underset{\sim}{m} = \underset{\sim}{M}^\alpha \nu_\alpha \quad , \quad \nu_\alpha = \underset{\sim}{a}_\alpha \cdot \underset{\sim}{\nu} \quad , \tag{6a,b,c}$$

where $\underset{\sim}{N}^\alpha$ and $\underset{\sim}{M}^\alpha$ are independent of $\underset{\sim}{\nu}$. Referring all tensor quantities to base vectors $\underset{\sim}{a}_i$, we may write**

$$\underset{\sim}{N}^\alpha = N^{i\alpha} \underset{\sim}{a}_i \quad , \quad \underset{\sim}{k} = k^i \underset{\sim}{a}_i \quad , \quad \underset{\sim}{M}^\alpha = M^{i\alpha} \underset{\sim}{a}_i \quad , \tag{7a,b,c}$$

where the usual summation convention is used.

Coupled thermomechanical constraints for a three-dimensional continuum have been previously discussed within the context of more classical thermodynamics [7] as well as within the context of recent developments [8]. Further, a rather general discussion of purely mechanical constraints for the theory of a Cosserat surface is contained

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*This notation differs from that used in [1,6]. In particular, we note that the quantities $\underset{\sim}{n}$, $\underset{\sim}{k}$, $\underset{\sim}{m}$ defined here, correspond, respectively to $\underset{\sim}{N}$, $\underset{\sim}{m}$, $\underset{\sim}{M}$ in [1,6].

**The notations used here for $\underset{\sim}{N}^\alpha$ and $\underset{\sim}{M}^\alpha$ are consistent with those of [2], but differ from the notation $\underset{\sim}{N}^\alpha = N^{\alpha i} \underset{\sim}{a}_i$ and $\underset{\sim}{M}^\alpha = M^{\alpha i} \underset{\sim}{a}_i$ used in [1,6].

in [9]. In these works, a constrained material is characterized by a set of constraint equations that restrict various kinematical quantities (deformation and temperature fields) as well as associated constraint responses that introduce a certain arbitrariness in the kinetic quantities.

For constrained theory III, we need introduce only a single constraint, which excludes normal extension and is characterized by

$$\underset{\sim}{d} \cdot \underset{\sim}{a}_3 = \underset{\sim}{D} \cdot \underset{\sim}{A}_3 = D \qquad (\gamma_3 = 0 \quad , \quad \dot{\gamma}_3 = 0) \tag{8}$$

Following previous works [6-8], we assume that the kinetic quantities* $\acute{N}^{\alpha\beta}$, k^{α}, $M^{i\alpha}$ are completely determined by constitutive equations and that k^3 separates into two additive parts: one part, denoted by $\hat{k}^3$, is determined by a constitutive equation; and the other part, $\bar{k}^3$, is an arbitrary function of (θ^{α},t), independent of strain rates, which is further assumed to be workless. Thus, we assume that

$$k^3 = \hat{k}^3 + \bar{k}^3 \quad , \quad \bar{k}^3 \, \dot{\gamma}_3 = 0 \quad . \tag{9a,b}$$

In view of the constraint (8), the director $\underset{\sim}{d}$ may be expressed in the form

$$\underset{\sim}{d} = d^{\alpha} \, \underset{\sim}{a}_{\alpha} + D \, \underset{\sim}{a}_3 \quad . \tag{10}$$

It is now convenient to record [1,2] the local forms of the basic equations of motion as:

$$\lambda = \rho \, a^{1/2} = \rho_o \, A^{1/2} \tag{11a}$$

$$\rho \, \hat{f}^{\alpha} + N^{\alpha\beta}|_{\beta} - N^{3\beta} \, b^{\alpha}_{\beta} = 0 \quad , \quad \rho \, \hat{f}^{3} + N^{3\beta}|_{\beta} + N^{\alpha\beta} \, b_{\alpha\beta} = 0 \quad , \tag{11b,c}$$

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*The quantity $\acute{N}^{\alpha\beta}$ will be defined presently.

$$\rho\, \hat{\ell}^{\alpha} - k^{\alpha} + M^{\alpha\beta}|_{\beta} - M^{3\beta}\, b^{\alpha}_{\beta} = 0 \quad , \tag{11d}$$

$$\bar{k}^{3} = \rho\, \hat{\ell}^{3} - \hat{k}^{3} + M^{3\beta}|_{\beta} + M^{\alpha\beta}\, b_{\alpha\beta} \quad , \tag{11e}$$

where

$$\acute{N}^{\alpha\beta} = \acute{N}^{\beta\alpha} = N^{\alpha\beta} - M^{\beta\sigma}\, \lambda^{\alpha}_{\cdot\sigma} - d^{\alpha}(\rho\, \hat{\ell}^{\beta} + M^{\alpha\sigma}|_{\sigma} - M^{3\sigma}\, b^{\beta}_{\sigma}) \tag{12a}$$

$$N^{3\alpha} = d^{3}(\rho\, \hat{\ell}^{\alpha} + M^{\alpha\beta}|_{\beta} - M^{3\beta}\, b^{\alpha}_{\beta}) - d^{\alpha}(\rho\, \hat{\ell}^{3} + M^{3\beta}|_{\beta} + M^{\alpha\beta}\, b_{\alpha\beta}) + \lambda^{3}_{\cdot\sigma}\, M^{\alpha\sigma} - \lambda^{\alpha}_{\cdot\sigma}\, M^{3\sigma} \tag{12b}$$

and where $\hat{f}^{i}$ and $\hat{\ell}^{i}$ are defined using the vectors $\underset{\sim}{a}^{i}$ which are the reciprocals of $\underset{\sim}{a}_{i}$, such that

$$\hat{f}^{i} = \underset{\sim}{a}^{i} \cdot (\underset{\sim}{f} - \dot{\underset{\sim}{v}} - y^{1}\, \dot{\underset{\sim}{w}}) \quad , \quad \hat{\ell}^{i} = \underset{\sim}{a}^{i} \cdot (\underset{\sim}{\ell} - y^{1}\, \dot{\underset{\sim}{v}} - y^{2}\, \dot{\underset{\sim}{w}}) \quad . \tag{13a,b}$$

In (11a) ρ_{o} is the reference value of the mass density ρ. Also, in (11) and (12) a bar denotes covariant differentiation with respect to the metric $a_{\alpha\beta}$. Equation (11a) represents conservation of mass, equations (11b,c) represent the balance of linear momentum, equations (11d,e) represent the balance of director momentum, and equations (12a,b) represent the balance of angular momentum.

Once appropriate constitutive equations are specified for the kinetic quantities

$$\{\acute{N}^{\alpha\beta}\ ,\ k^{\alpha},\ M^{i\alpha}\} \quad , \tag{14}$$

the six equations (11a-d) may be used to determine the six unknowns

$$\{\rho,\ \underset{\sim}{r},\ d^{\alpha}\} \quad . \tag{15}$$

For an arbitrary value of $\hat{k}^{3}$, equation (11e) may be satisfied by an appropriate specification of $\bar{k}^{3}$. Now, equations (11b-d) must be solved

subject to certain boundary and initial conditions, which are discussed in the next section.

Boundary and Initial Conditions

A rather detailed discussion of boundary conditions for constrained theories I and II is included in [1, Sec. 15] and [2], respectively. Basically, we follow the discussion in [1] and recall [2] that the boundary integral appearing in the energy equation takes the form

$$\int_{\partial P} (\underset{\sim}{n} \cdot \underset{\sim}{v} + \underset{\sim}{m} \cdot \underset{\sim}{w}) ds = \int_{\partial P} (n^i \, v_i + m^i \, w_i) ds \quad , \tag{16}$$

where ds is the elemental arclength of ∂P and where

$$n^i = \underset{\sim}{a}^i \cdot \underset{\sim}{n} = N^{i\alpha} \, \nu_\alpha \quad , \quad m^i = \underset{\sim}{a}^i \cdot \underset{\sim}{m} = M^{i\alpha} \, \nu_\alpha \tag{17a,b,c}$$

$$v_i = \underset{\sim}{a}_i \cdot \underset{\sim}{v} \quad , \quad w_i = \underset{\sim}{w} \cdot \underset{\sim}{a}_i \quad . \tag{17d,e}$$

Recalling [1], that an arbitrary function $F = F(\theta^\alpha, t)$ may be expressed in terms of its normal and tangential derivatives $\partial F/\partial \nu$ and $\partial F/\partial s$, respectively, we may integrate (16) by parts, assume continuity on ∂P, and use (10) to rewrite (16) in the form

$$\int_{\partial P} \Big[P^i(v_i) + m^\sigma\Big(\dot{d}_\sigma - \nu_\sigma \, d^\beta \frac{\partial v_\beta}{\partial \nu} - \nu_\sigma \, D \, \frac{\partial v_3}{\partial \nu}\Big) + m^3\Big(d^\beta \, \nu_\beta \, \frac{\partial v_3}{\partial \nu}\Big)\Big] ds \tag{18}$$

where P^i are given by

$$P^\sigma = \big[n^\sigma + m^\beta(d^\lambda \, \Gamma^\sigma_{\lambda\beta} - D \, b^\sigma_\beta) + m^3 \, d^\beta \, b^\sigma_\beta\big] + \frac{\partial}{\partial s}\big(m^\beta \, d^\sigma \, \lambda_\beta\big) \quad , \tag{19a}$$

$$P^3 = (n^3 + m^\beta \, d^\sigma \, b_{\sigma\beta}) + \frac{\partial}{\partial s}\big[(m^\beta \, D - m^3 \, d^\beta)\lambda_\beta\big] \quad , \tag{19b}$$

and where the unit tangent vector $\underset{\sim}{\lambda}$ to ∂P and the Christoffel symbol $\Gamma^\sigma_{\lambda\beta}$ are defined by

$$\underset{\sim}{\lambda} = \underset{\sim}{a}_3 \times \underset{\sim}{\nu} \quad , \quad \Gamma^{\sigma}_{\lambda\beta} = \underset{\sim}{a}_{\lambda,\beta} \cdot \underset{\sim}{a}^{\sigma} \quad . \tag{20a,b}$$

It is clear from (18) that at each point of ∂P, we must specify either the kinetic variables (p^i, m^i) or their associated kinematic variables that are written in parentheses in (18). Thus, for the nonlinear constrained theory III, we must specify six boundary conditions at each point of ∂P. However, for a theory that is linearized about the reference configuration, the quantities p^i, m^i, d_σ, v_i are small and of order $\varepsilon(o < \varepsilon << 1)$ so that the term in (18) associated with m^3 is negligible compared to the other terms. Thus, for the linearized theory the number of boundary conditions reduces to five.

To see that the terms in (18) associated with bending of a plate are consistent with those discussed in [5], we note that for the linear theory of a plate

$$p^{\sigma} = n^{\sigma} \quad , \quad p^3 = n^3 + \frac{\partial}{\partial s}(m^{\sigma}\lambda_{\sigma} D) \quad , \tag{21a,b}$$

$$\underset{\sim}{w} = (\dot{d}_{\sigma} - D\, v_{3,\sigma})\underset{\sim}{A}^{\sigma} = \left(\dot{d}_s - D\frac{\partial v_3}{\partial s}\right)\underset{\sim}{\lambda} + \left(\dot{d}_{\nu} - D\frac{\partial v_3}{\partial \nu}\right)\underset{\sim}{\nu} \quad , \tag{21c}$$

$$d_s = \underset{\sim}{\lambda} \cdot \underset{\sim}{d} \quad , \quad d_{\nu} = \underset{\sim}{\nu} \cdot \underset{\sim}{d} \quad , \tag{21d,e}$$

where we have introduced the temporary notation d_s and d_ν, respectively, for the tangential and normal components of $\underset{\sim}{d}$ relative to the curve ∂P. Thus, for a plate, (18) may be rewritten as

$$\int_{\partial P} \left[n^i v_i + \frac{\partial}{\partial s}(m^{\sigma}\lambda_{\sigma} D\, v_3) + m^{\sigma}\left(\dot{d}_{\sigma} - D\,\lambda_{\sigma}\frac{\partial v_3}{\partial s} - D\,\nu_{\sigma}\frac{\partial v_3}{\partial \nu}\right)\right]ds \tag{22}$$

Integrating the second term in (22) and using continuity, we may rewrite (22) in the form

$$\begin{aligned}\int_{\partial P} \Big[&(\underset{\sim}{n}\cdot\underset{\sim}{\lambda})(\underset{\sim}{v}\cdot\underset{\sim}{\lambda}) + (\underset{\sim}{n}\cdot\underset{\sim}{\nu})(\underset{\sim}{v}\cdot\underset{\sim}{\nu}) + n^3 v_3 \\ &+ (\underset{\sim}{m}\cdot\underset{\sim}{\lambda})\left(\dot{d}_s - D\frac{\partial v_3}{\partial s}\right) + (\underset{\sim}{m}\cdot\underset{\sim}{\nu})\left(\dot{d}_{\nu} - D\frac{\partial v_3}{\partial \nu}\right)\Big]ds\end{aligned} \tag{23}$$

Next, recall [1] that the Cosserat theory may be brought into a one-to-one correspondence with the three-dimensional theory if the position vector $\underset{\sim}{p}^*$ locating points in the shell admits the representation

$$\underset{\sim}{p}^* = \underset{\sim}{r} + \theta^3 \underset{\sim}{d} \tag{24}$$

where θ^3 is the convected coordinate through the thickness of the shell. Using the notation $\bar{u}'_s$ and $\bar{u}'_n$ defined in [5], it follows from (21c) and (24) that $\underset{\sim}{w} \cdot \underset{\sim}{\lambda}$ and $\underset{\sim}{w} \cdot \underset{\sim}{\nu}$ correspond to $\bar{u}'_s$ and $\bar{u}'_n$, respectively, and that the last three terms in (23) correspond to a dynamical version of the boundary integral in [5], where we note that $D = 1$ characterizes a shell of constant thickness.

Finally, for a dynamical problem it is necessary to specify initial conditions of the form

$$\underset{\sim}{r} = \bar{\underset{\sim}{r}}(\theta^\alpha) \quad , \quad \underset{\sim}{v} = \bar{\underset{\sim}{v}}(\theta^\alpha) \quad , \quad \underset{\sim}{d} = \bar{\underset{\sim}{d}}(\theta^\alpha) \quad , \quad \underset{\sim}{w} = \bar{\underset{\sim}{w}}(\theta^\alpha) \quad , \quad \text{for } t = 0 \tag{25a,b,cd}$$

where $\bar{\underset{\sim}{r}}$, $\bar{\underset{\sim}{v}}$, $\bar{\underset{\sim}{d}}$, $\bar{\underset{\sim}{w}}$ are specified functions of θ^α only and $\bar{\underset{\sim}{d}}$, $\bar{\underset{\sim}{w}}$ are consistent with the constrained form (10) of the director.

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Appendix C

ON THE DETERMINATION OF CERTAIN CONSTITUTIVE COEFFICIENTS FOR THERMOELASTIC SHELLS

(Submitted for publication to the International Journal of Solids and Structures)

ABSTRACT

In this paper we model a shell composed of a linear elastic, homogeneous, isotropic material as a Cosserat surface. Specific attention is focused on the determination of certain thermal and mechanical constitutive coefficients, which were previously determined by integrating three-dimensional constitutive equations. Here, we determine these coefficients by comparing Cosserat solutions with exact three-dimensional solutions. This comparison suggests values for one of the thermal coefficients and two of the mechanical coefficients that are different from those previously proposed [1,2].

1. Introduction

Classical developments of shell theory usually start with a complete three-dimensional description of a shell-like body including constitutive equations. Shell equations that are functions of two space-variables and time are then developed by introducing approximations and either integrating the equations of motion through the thickness of the shell or by using an integral form of a variational principle. More recently, it has become common to develop shell equations by a direct approach in which the shell is modeled as a Cosserat surface [1]. This latter approach has the distinct advantage over the classical approach that equations can be developed for shells with arbitrary constitutive properties. More specifically, the discussion of constitutive equations within the context of the Cosserat theory is very similar to that in the three-dimensional theory. Constitutive coefficients are determined by comparing the predictions of shell theory with experimental data or exact solutions of the three-dimensional equations that have already been shown to accurately predict experimental data.

Until recently, the Cosserat theory admitted any finite number of directors to provide information about the variation of mechanical variables through the thickness of the shell but only admitted a single temperature field to model the average temperature in the shell. Recent advances in thermodynamics have allowed the Cosserat theory to be generalized to admit any finite number of temperature fields that provide information about the variation of temperature through the thickness of the shell [2]. An important special case of the general theory is one that admits a single director and two temperature fields.

In [2] specific constitutive equations are developed for a plate composed of a linear elastic, homogeneous, isotropic material. Values for most of the mechanical constitutive coefficients and for some of the thermal coefficients have been determined by comparing predictions of the Cosserat theory with exact solutions of the three-dimensional equations [1]. Values for the mechanical constitutive coefficients

associated with tangential shear deformation (more commonly referred to as transverse shear deformation) and the new thermal coefficients were determined by integrating three-dimensional constitutive equations [1,2]. Even though a one-to-one correspondence between the three-dimensional theory and the Cosserat theory may be established, these two methods of determining constitutive coefficients do not always yield the same results (see additional comments in Sections 4 and 5).

The objective of this paper is to show that the new thermal coefficients and the mechanical coefficients associated with tangential shear deformation may be determined by direct comparison with exact three-dimensional solutions. In the following sections, we record the basic equations valid for a shell composed of a linear elastic, homogeneous, isotropic material. Next, we briefly recall [1] how most of the mechanical coefficients were determined by comparison with exact solutions. Then, we determine the new thermal coefficients by comparing with simple solutions of the three-dimensional heat conduction equation for a rigid plate. One of the thermal coefficients determined in this manner has a value different from that proposed in [2]. Finally, we determine the two mechanical coefficients associated with tangential shear deformation by comparing with the exact solutions of simple shear of a plate and twisting of a circular cylindrical shell. The values of both of these coefficients determined in this manner are different from those proposed in [1].

2. Basic Equations

Let the material points of the Cosserat surface C be identified by means of a system of convected coordinate $\theta^{\alpha}(\alpha = 1,2)$ and let the two-dimensional region of space occupied by the material surface in the present configuration at time t be denoted by c. Further, let the vector valued function $\underset{\sim}{x}$ define the position of a material point of the surface C and at each such point define the vector valued function $\underset{\sim}{d}$, called the director, and the two temperature fields θ and ϕ, each referred to the present configuration. Then, a thermomechanical process

of the Cosserat surface is defined by

$$\underset{\sim}{r} = \underset{\sim}{r}(\theta^\alpha, t) \quad , \quad \underset{\sim}{d} = \underset{\sim}{d}(\theta^\alpha, t) \quad , \quad [\underset{\sim}{a}_1 \; \underset{\sim}{a}_2 \; \underset{\sim}{d}] > 0 \quad , \tag{2.1a,b,c}$$

$$\theta = \theta(\theta^\alpha, t) \quad , \quad (\theta > 0) \quad , \quad \phi = \phi(\theta^\alpha, t) \quad , \tag{2.1d,e,f}$$

where the tangent vectors $\underset{\sim}{a}_\alpha$ and the unit normal vector $\underset{\sim}{a}_3$ are defined by

$$\underset{\sim}{a}_\alpha = \frac{\partial \underset{\sim}{r}}{\partial \theta^\alpha} \quad , \quad \underset{\sim}{a}_\alpha \cdot \underset{\sim}{a}_3 = 0 \quad , \quad \underset{\sim}{a}_3 \cdot \underset{\sim}{a}_3 = 1 \quad , \quad a^{1/2} = [\underset{\sim}{a}_1 \; \underset{\sim}{a}_2 \; \underset{\sim}{a}_3] > 0 \quad , \tag{2.2a,b,c,d}$$

and the condition (2.1c) ensures that the director is nowhere tangent to c. The velocity $\underset{\sim}{v}$ and director velocity $\underset{\sim}{w}$ may be defined by

$$\underset{\sim}{v} = \dot{\underset{\sim}{r}} \quad , \quad \underset{\sim}{w} = \dot{\underset{\sim}{d}} \quad , \tag{2.3a,b}$$

where a superposed dot denotes time differentiation holding θ^α fixed. In the reference configuration, we assume that the shell has uniform thickness h and is at uniform temperature θ_o. Then, the reference values of the various kinematic quantities may be denoted by*

$$\underset{\sim}{r} = \underset{\sim}{R} \quad , \quad \underset{\sim}{d} = \underset{\sim}{D} = \underset{\sim}{A}_3 \quad , \quad \underset{\sim}{a}_i = \underset{\sim}{A}_i \quad , \quad a^{1/2} = A^{1/2} \quad , \tag{2.4a,b,c,d}$$

$$\theta = \theta_o \quad , \quad \phi = 0 \quad , \tag{2.4e,f}$$

where $\underset{\sim}{R}$, $\underset{\sim}{A}_i$ and $A^{1/2}$ depend on the coordinates θ^α only.

Let P, bounded by the closed curve ∂P, denote the region occupied by an arbitrary material portion of the surface c in the present

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*Throughout the text Greek indices have a range (1,2) and Latin indices have a range (1,2,3).

configuration and let $\underset{\sim}{\nu}$ be the unit outward normal to ∂P. Using the notation* of [3] and referring to the present configuration, we define the following quantities: the positive mass density (mass per unit area of P) $\rho = \rho(\theta^\alpha, t)$; the contact force $\underset{\sim}{n} = \underset{\sim}{n}(\theta^\alpha, t; \underset{\sim}{\nu})$ and the contact director couple $\underset{\sim}{m} = \underset{\sim}{m}(\theta^\alpha, t; \underset{\sim}{\nu})$, each per unit length of the curve ∂P; the specific (per unit mass of P) assigned force $\underset{\sim}{f} = \underset{\sim}{f}(\theta^\alpha, t)$ and specific assigned director couple $\underset{\sim}{\ell} = \underset{\sim}{\ell}(\theta^\alpha, t)$; the intrinsic director couple $\underset{\sim}{k} = \underset{\sim}{k}(\theta^\alpha, t)$ per unit area of P; the inertia coefficients $y^1 = y^1(\theta^\alpha)$ and $y^2 = y^2(\theta^\alpha)$ which are independent of time; the specific entropies $\eta = \eta(\theta^\alpha, t)$ and $\eta_1 = \eta_1(\theta^\alpha, t)$; the specific internal rates of production of entropy $\xi = \xi(\theta^\alpha, t)$, $\xi_1 = \xi_1(\theta^\alpha, t)$, and $\bar{\xi}_1 = \bar{\xi}_1(\theta^\alpha, t)$; the entropy fluxes $k = k(\theta^\alpha, t; \underset{\sim}{\nu})$ and $k_1 = k_1(\theta^\alpha, t; \underset{\sim}{\nu})$; the specific external rates of supply of entropy $s = s(\theta^\alpha, t)$ and $s_1 = s_1(\theta^\alpha, t)$; the specific internal energy $\varepsilon = \varepsilon(\theta^\alpha, t)$; and the specific Helmholtz free energy $\psi = \psi(\theta^\alpha, t) \equiv \varepsilon - \theta\eta - \phi\eta_1$.

For the linear theory, it is convenient to introduce the displacement vector $\underset{\sim}{u}$ and director displacement $\underset{\sim}{\delta}$ relative to the reference configuration by the equations

$$\underset{\sim}{r} = \underset{\sim}{R} + \underset{\sim}{u} \quad , \quad \underset{\sim}{d} = \underset{\sim}{D} + \underset{\sim}{\delta} \quad . \tag{2.5a,b}$$

We now assume that in the present configuration, the displacements $\underset{\sim}{u}$, $\underset{\sim}{\delta}$ and the temperatures $(\theta - \theta_o)$ and ϕ are of order** $\varepsilon (0 < \varepsilon \ll 1)$ and that quadratic terms in these quantities may be neglected relative to linear terms. With this background, we record [1] expressions for the kinematic quantities $A_{\alpha\beta}$, $B_{\alpha\beta}$, $\Lambda_{i\alpha}$, and for the strains $e_{\alpha\beta}$, γ_i, $\kappa_{i\alpha}$ in the forms

*This notation differs from that used in [1,2].

**The temporary use of this symbol for the small parameter should not be confused with the use of the same symbol elsewhere for the internal energy.

$$A_{\alpha\beta} = \underset{\sim}{A}_\alpha \cdot \underset{\sim}{A}_\beta \quad , \quad B_{\alpha\beta} = \underset{\sim}{A}_{\alpha,\beta} \cdot \underset{\sim}{A}_3 \quad , \tag{2.6a,b}$$

$$\Lambda_{\alpha\beta} = -B_{\alpha\beta} \quad , \quad \Lambda_{3\alpha} = 0 \quad , \tag{2.6c,d,}$$

$$e_{\alpha\beta} = \frac{1}{2}(u_{\alpha|\beta} + u_{\beta|\alpha}) - B_{\alpha\beta}\, u_3 \quad , \tag{2.6e}$$

$$\gamma_\alpha = \delta_\alpha + u_{3,\alpha} + B_\alpha^\lambda\, u_\lambda \quad , \quad \gamma_3 = \delta_3 \quad , \tag{2.6f,g}$$

$$\kappa_{\alpha\beta} = \rho_{\alpha\beta} - B_{\alpha\beta}\,\gamma_3 \quad , \quad \kappa_{3\alpha} = \rho_{3\alpha} + B_\alpha^\lambda\,\gamma_\lambda \quad , \tag{2.6h,i}$$

$$\rho_{\alpha\beta} = -u_{3|\alpha\beta} - B^\lambda_{\alpha|\beta}\, u_\lambda - B^\lambda_\alpha\, u_{\lambda|\beta} - B^\lambda_\beta\, u_{\lambda|\alpha} + B_{\alpha\lambda}\, B^\lambda_\beta\, u_3 + \gamma_{\alpha|\beta} \quad , \tag{2.6j}$$

$$\rho_{3\alpha} = \gamma_{3,\alpha} \quad , \tag{2.6k}$$

where for the linear theory all tensor quantities are referred to the base vectors $\underset{\sim}{A}_i$. In (2.6) and through the text, we use the usual summation convention over repeated indices; a comma denotes partial differentiation with respect to θ^α, and a bar denotes covariant differentiation with respect to the metric $A_{\alpha\beta}$.

With suitable continuity assumptions, it can be shown that* [1,2]

$$\underset{\sim}{n} = \underset{\sim}{N}^\alpha\, \nu_\alpha = (N^{i\alpha}\, \underset{\sim}{A}_i)\nu_\alpha \quad , \quad \underset{\sim}{m} = \underset{\sim}{M}^\alpha\, \nu_\alpha = (M^{i\alpha}\, \underset{\sim}{A}_i)\nu_\alpha \quad , \tag{2.7a,b}$$

$$k = \underset{\sim}{p} \cdot \underset{\sim}{\nu} = p^\alpha\, \nu_\alpha \quad , \quad k_1 = \underset{\sim}{p}_1 \cdot \underset{\sim}{\nu} = p_1^\alpha\, \nu_\alpha \quad , \tag{2.7c,d}$$

where $\nu_\alpha = \underset{\sim}{A}_\alpha \cdot \underset{\sim}{\nu}$ are the components of the normal vector $\underset{\sim}{\nu}$ and where $\underset{\sim}{N}^\alpha$, $\underset{\sim}{M}^\alpha$, $\underset{\sim}{p}$, and $\underset{\sim}{p}_1$ are independent of $\underset{\sim}{\nu}$. Further, with reference to the energy equation, the specific external rates of heat supply r and r_1, and the heat flux vectors $\underset{\sim}{q}$ and $\underset{\sim}{q}_1$ are defined by

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*The notation in (2.7a,b) is consistent with that in [3], but differs from that in [1,2].

$$r = \theta s \quad , \quad r_1 = \phi s_1 \quad , \quad \underset{\sim}{q} = \theta \underset{\sim}{p} \quad , \quad \underset{\sim}{q}_1 = \phi \underset{\sim}{p}_1 \quad . \tag{2.8a,b,c,d}$$

Now the local forms of the basic equations may be recorded as

$$\lambda = \rho\, a^{1/2} = \rho_o\, A^{1/2} \tag{2.9a}$$

$$\rho_o\, \hat{f}^\alpha + N^{\alpha\beta}|_\beta - N^{3\beta}\, B^\alpha_\beta = 0 \quad , \quad \rho_o\, \hat{f}^3 + N^{3\beta}|_\beta + N^{\alpha\beta}\, B_{\alpha\beta} = 0 \quad , \tag{2.9b,c}$$

$$\rho_o\, \hat{\ell}^\alpha - k^\alpha + M^{\alpha\beta}|_\beta - M^{3\beta}\, B^\alpha_\beta = 0 \quad , \quad \rho_o\, \hat{\ell}^3 - k^3 + M^{3\beta}|_\beta + M^{\alpha\beta}\, B_{\alpha\beta} = 0 \quad , \tag{2.9d,e}$$

$$\rho_o\, \dot{\eta} = \rho_o\, s - p^\alpha|_\alpha \quad , \quad \rho_o\, \dot{\eta}_1 = \rho_o(s_1 + \xi_1) - p_1^\alpha|_\alpha \quad , \tag{2.9f,g}$$

where

$$\acute{N}^{\alpha\beta} = \acute{N}^{\beta\alpha} = N^{\alpha\beta} + M^{\beta\sigma}\, B^\alpha_\sigma \quad , \quad N^{\alpha 3} = \rho_o\, \hat{\ell}^\alpha + M^{\alpha\beta}|_\beta \quad , \tag{2.10a,b}$$

and where

$$\hat{f}^i = \underset{\sim}{A}^i \cdot (\underset{\sim}{f} - \dot{\underset{\sim}{v}} - y^1\, \dot{\underset{\sim}{w}}) \quad , \quad \hat{\ell}^i = A^i \cdot (\underset{\sim}{\ell} - y^1\, \dot{\underset{\sim}{v}} - y^2\, \dot{\underset{\sim}{w}}) \quad . \tag{2.11a,b}$$

In (2.9) ρ_o is the reference value of ρ and in (2.11) $\underset{\sim}{A}^i$ are the reciprocal vectors of $\underset{\sim}{A}_i$. Equation (2.9a) represents conservation of mass, (2.9b,c) represent the balance of linear momentum, (2.9d,e) represent the balance of director momentum, (2.9f,g) represent balances of entropy, and (2.10a,b) represent balance of angular momentum. These equations must be supplemented by an energy equation and constitutive equations. It was shown in [2] that the energy equation for a linear thermoelastic shell is satisfied provided that

$$\acute{N}^{\alpha\beta} = \rho_o \frac{\partial\psi}{\partial e_{\alpha\beta}} \quad , \quad k^i = \rho_o \frac{\partial\psi}{\partial\gamma_i} \quad , \quad M^{i\alpha} = \rho_o \frac{\partial\psi}{\partial\kappa_{i\alpha}} \quad , \tag{2.12a,b,c}$$

$$\eta = -\frac{\partial\psi}{\partial\theta} \quad , \quad \eta_1 = -\frac{\partial\psi}{\partial\phi} \quad . \tag{2.12d,e}$$

Additional terms appear in the energy equation, but they are of higher order and thus are neglected for the linear theory.

Now we confine attention to a thermoelastic that is isotropic in its reference configuration and specify linear constitutive equations of the form

$$2\,\rho_o\,\phi = [\alpha_1\,A^{\alpha\beta}\,A^{\gamma\delta} + \alpha_2(A^{\alpha\gamma}\,A^{\beta\delta} + A^{\alpha\delta}\,A^{\beta\gamma})]e_{\alpha\beta}\,e_{\gamma\delta}$$
$$+ \alpha_4(\gamma_3)^2 + [\alpha_5\,A^{\alpha\beta}\,A^{\gamma\delta} + \alpha_6(A^{\alpha\gamma}\,A^{\beta\delta} + A^{\alpha\delta}\,A^{\beta\gamma})]\kappa_{\alpha\beta}\,\kappa_{\gamma\delta}$$
$$+ \alpha_3\,A^{\alpha\beta}\,\gamma_\alpha\,\gamma_\beta + \alpha_8\,A^{\alpha\beta}\,\kappa_{3\alpha}\,\kappa_{3\beta} + 2\,\alpha_9\,A^{\alpha\beta}\,e_{\alpha\beta}\,\gamma_3$$
$$- 2\,\beta_o\,\gamma_3(\theta - \theta_o) - 2\,\beta_1\,A^{\alpha\beta}\,e_{\alpha\beta}(\theta - \theta_o) - 2\,\beta_2\,A^{\alpha\beta}\,\kappa_{\alpha\beta}\,\phi$$
$$- \beta_3(\theta^2 - 2\,\theta\,\theta_o) - \beta_4\,\phi^2 - 2\,\beta_5\,\theta \qquad (2.13a)$$

$$\underset{\sim}{p} = -\,a_o\,\underset{\sim}{g} \quad , \quad \underset{\sim}{p}_1 = -\,b_1\,\underset{\sim}{g}_1 \quad , \qquad (2.13b,c)$$

$$\xi = 0 \quad , \quad \rho_o\,\xi_1 = \rho_o\,\bar{\xi}_1 = -\,b_2\,\phi \quad , \qquad (2.13d,e)$$

where the coefficients α_1-α_6, α_8, α_9, β_o-β_5, a_o, b_1, b_2 in (2.13) are constants and where the temperature gradients $\underset{\sim}{g}$ and $\underset{\sim}{g}_1$ are defined by

$$\underset{\sim}{g} = \theta_{,\alpha}\,\underset{\sim}{A}^\alpha \quad , \quad \underset{\sim}{g}_1 = \phi_{,\alpha}\,\underset{\sim}{A}^\alpha \quad . \qquad (2.14a,b)$$

The constitutive assumption in (2.13a) is slightly different from that used in [2], with the main difference being that here ϕ is a function of $\kappa_{1\alpha}$ instead of the kinematic variable $\rho_{1\alpha}$ (see additional comments in Section 5). It follows from (2.12) and (2.13) that

$$\acute{N}^{\alpha\beta} = [\alpha_1\,A^{\alpha\beta}\,A^{\gamma\delta} + \alpha_2(A^{\alpha\gamma}\,A^{\beta\delta} + A^{\alpha\delta}\,A^{\beta\gamma})]e_{\gamma\delta}$$
$$+ \alpha_9\,A^{\alpha\beta}\,\gamma_3 - \beta_1\,A^{\alpha\beta}(\theta - \theta_o) \quad , \qquad (2.15a)$$

$$k^{\alpha} = \alpha_3 A^{\alpha\beta} \gamma_{\beta} \quad , \quad k^3 = \alpha_4 \gamma_3 + \alpha_9 A^{\alpha\beta} e_{\alpha\beta} - \beta_o(\theta - \theta_o) \quad , \tag{2.15b,c}$$

$$M^{\alpha\beta} = [\alpha_5 A^{\alpha\beta} A^{\gamma\delta} + \alpha_6(A^{\alpha\gamma} A^{\beta\delta} + A^{\alpha\delta} A^{\beta\gamma})]\kappa_{\gamma\delta} - \beta_2 A^{\alpha\beta} \phi \quad , \tag{2.15d}$$

$$M^{3\alpha} = \alpha_8 A^{\alpha\beta} \kappa_{3\beta} \quad , \tag{2.15e}$$

$$\rho_o \eta = \beta_o \gamma_3 + \beta_1 A^{\alpha\beta} e_{\alpha\beta} + \beta_3(\theta - \theta_o) + \beta_5 \quad , \tag{2.15f}$$

$$\rho_o \eta_1 = \beta_2 A^{\alpha\beta} \kappa_{\alpha\beta} + \beta_4 \phi \quad . \tag{2.15g}$$

In general, these constitutive equations must be further restricted by statements of the second law of thermodynamics [2]. For the thermoelastic shell considered here, these restrictions reduce to

$$a_o \geq 0 \quad , \quad b_1 \geq 0 \quad , \quad b_2 \geq 0 \quad , \quad \beta_3 > 0 \quad . \tag{2.16a,b,c,d}$$

Before closing this section, we recall [1,2] that this theory, which is developed by direct approach, may be brought into a one-to-one correspondence with the three-dimensional theory by assuming that the position vector $\underset{\sim}{r}^*$ of a point in the shell and the temperature field θ^* admit the representations

$$\underset{\sim}{r}^* = \underset{\sim}{r}^*(\theta^{\alpha},\theta^3,t) = \underset{\sim}{r}(\theta^{\alpha},t) + \theta^3 \underset{\sim}{d}(\theta^{\alpha},t) \quad , \tag{2.17a}$$

$$\theta^* = \theta^*(\theta^{\alpha},\theta^3,t) = \theta(\theta^{\alpha},t) + \theta^3 \phi(\theta^{\alpha},t) \quad , \tag{2.17b}$$

where θ^3 is a coordinate through the thickness of the shell. For a shell of constant thickness h, we may choose the reference surface of the shell to be the middle surface and define the top surface ∂P^+ of the shell by $\theta^3 = h/2$ and the bottom surface ∂P^- by $\theta^3 = -h/2$. With reference to the three-dimensional theory, we recall [2] the definitions*

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*Our use of the symbol $\underset{\sim}{g}_1$ for the base vector should not be confused with the use of the same symbol for the temperature gradient in (2.14b).

$$g_i = \frac{\partial \underset{\sim}{r}^*}{\partial \theta^i} \ , \quad g_i \cdot g^j = \delta_i^j \ , \quad g^{1/2} = [g_1 \ g_2 \ g_3] > 0 \ , \tag{2.18a,b,c}$$

$$g_{ij} = g_i \cdot g_j \ , \quad g^{ij} = g^i \cdot g^j \ , \tag{2.18d,e}$$

where g_i and g^i are, respectively, the base vectors and their reciprocals in the present configuration and where δ_i^j is the Kronecker symbol. We also recall [2] the following relationships for the linear theory

$$\lambda = \int_{-h/2}^{h/2} \rho_o^* \, G^{1/2} \, d\theta^3 \ , \tag{2.19a}$$

$$\lambda y^N = \int_{-h/2}^{h/2} \rho_o^* \, G^{1/2} (\theta^3)^N \, d\theta^3 \qquad (N = 1,2) \ , \tag{2.19b}$$

$$\lambda \underset{\sim}{f} = \int_{-h/2}^{h/2} \rho_o^* \, G^{1/2} \, \underset{\sim}{f}^* \, d\theta^3 + B^+ \, \underset{\sim}{t}^+ + B^- \, \underset{\sim}{t}^- \ , \tag{2.19c}$$

$$\lambda \underset{\sim}{\ell} = \int_{-h/2}^{h/2} \rho_o^* \, G^{1/2} \, \underset{\sim}{f}^* \, \theta^3 \, d\theta^3 + \left(\frac{h}{2}\right) B^+ \, \underset{\sim}{t}^+ - \left(\frac{h}{3}\right) B^- \, \underset{\sim}{t}^- \ , \tag{2.19d}$$

$$\lambda s = \int_{-h/a}^{h/2} \rho_o^* \, G^{1/2} \, s^* \, d\theta^3 - B^+ \, k^+ - B^- \, k^- \ , \tag{2.9e}$$

$$\lambda s_1 = \int_{-h/a}^{h/2} \rho_o^* \, G^{1/2} \, s^* \, \theta^3 \, d\theta^3 - \left(\frac{h}{2}\right) B^+ \, k^+ + \left(\frac{h}{2}\right) B^- \, k^- \ , \tag{2.19f}$$

where ρ_o^*, $\underset{\sim}{f}^*$, s^* are, respectively, the three-dimensional mass density (mass per unit volume in the reference configuration), specific body force, and specific rate of entropy supply; $\underset{\sim}{t}^+$ and $\underset{\sim}{t}^-$ are, respectively, the surface tractions applied to the major surface ∂P^+ and ∂P^-; k^+ and k^- are, respectively, the entropy fluxes applied to the major surfaces ∂P^+ and ∂P^-; and $G^{1/2}$ is the reference value of $g^{1/2}$. It follows from (2.4), (2.17), and (2.18) that

$$G^{1/2} = A^{1/2}[1 - \theta^3 B^\sigma_\sigma + (\theta^3)^2 (B^1_1 B^2_2 - B^2_1 B^1_2)] \tag{2.20}$$

and from [2] that the functions B^+ and B^- take the values

$$B^+ = A^{1/2}[1 - \frac{h}{2} B^\sigma_\sigma + \frac{h^2}{4} (B^1_1 B^2_2 - B^2_1 B^1_2)] \quad , \tag{2.21a}$$

$$B^- = A^{1/2}[1 + \frac{h}{2} B^\sigma_\sigma + \frac{h^2}{4} (B^1_1 B^2_2 - B^2_1 B^1_2)] \quad . \tag{2.21b}$$

Also, we note that for the linear theory the quantities $\theta_o k^+$ and $\theta_o k^-$ represent, respectively, the heat fluxes measured positive for heat flowing out of the surfaces ∂P^+ and ∂P^-.

3. Determination of Constitutive Coefficients

In this section, we mainly recall results that were obtained in [1] for most of the mechanical constitutive coefficients. Confining attention to a shell of uniform thickness h and uniform density ρ^*_o in its reference configuration, we may substitute (2.20) into equations (2.19a,b) to obtain the expressions

$$\lambda = \rho_o A^{1/2} = (\rho^*_o h A^{1/2})[1 + \frac{h^2}{12} (B^1_1 B^2_2 - B^2_1 B^1_2)] \quad , \tag{3.1a}$$

$$\lambda y^1 = - (\rho^*_o h A^{1/2})(\frac{h^2}{12} B^\sigma_\sigma) \quad , \tag{3.1b}$$

$$\lambda y^2 = (\rho^*_o h A^{1/2})(\frac{h^2}{12})[1 + \frac{3 h^2}{20} (B^1_1 B^2_2 - B^2_1 B^1_2)] \quad , \tag{3.1c}$$

which determine the reference density ρ_o and the inertia coefficients y^1, y^2.

Values for most of the constitutive coefficients were obtained in [1] by comparing results of the Cosserat theory with exact three-dimensional results. Even though the constitutive equations (2.13) and (2.15) are postulated for shells, it suffices to evaluate most of the constitutive coefficients by considering solutions of plate problems. By solving the isothermal problem of a plate subjected to uniaxial

stress (or resultant force) and that for a plate subjected to hydrostatic pressure it may be shown [1] that

$$\alpha_1 = \alpha_9 = \frac{\nu(1 - \nu)}{(1 - 2\nu)} C \quad , \quad \alpha_2 = \left(\frac{1 - \nu}{2}\right) C = \mu h \quad , \tag{3.2a,b}$$

$$\alpha_4 = \frac{(1 - \nu)^2}{(1 - 2\nu)} C \quad , \quad C = \frac{Eh}{1 - \nu^2} \quad , \tag{3.2c,d}$$

where E is Young's modulus, ν is Poisson's ratio, and μ is one of Lamé's constants, all associated with the three-dimensional material. Similarly, the isothermal problem of pure bending of a plate may be solved to obtain the results

$$\alpha_5 = \nu B \quad , \quad \alpha_6 = \left(\frac{1 - \nu}{2}\right) B \quad , \quad B = \frac{Eh^3}{12(1 - \nu^2)} \quad . \tag{3.3a,b,c}$$

The thermal coefficients and the remaining mechanical coefficients will be determined in the next two sections.

4. Determination of the Thermal Coefficients

To determine the thermoelastic coefficients β_o, β_1, β_2, we follow [1] and consider the problems of free thermal expansion of a plate and free thermal bending to obtain

$$\beta_o = \beta_1 = \frac{Eh\ \alpha^*}{1 - 2\nu} \quad , \quad \beta_2 = \frac{Eh^3\ \alpha^*}{12(1 - \nu)} \quad , \tag{4.1a,b}$$

where α^* is the coefficient of linear thermal expansion associated with the three-dimensional material.

The remaining thermal coefficients were determined in [2] by direct integration of the constitutive equations. Here, instead, we determine these coefficients by comparison with exact solutions of a rigid heat conducting plate. Specifically, we consider the problem of a plate that is initially at uniform temperature θ_o and that is subjected to a uniform heat flux q^+ on its top surface, zero heat flux on its bottom surface, and no external supply of heat ($s^* = 0$). The quantity q^+ is

taken to be positive when heat flows out of the plate. Using $k^+ = q^+/\theta_o$ and $k^- = 0$, the solution of equations (2.9f,g) may be written in the form

$$\theta = \theta_o - \left(\frac{q^+}{\beta_3\ \theta_o}\right)t \quad , \quad \phi = -\left(\frac{hq^+}{2b_2\ \theta_o}\right)\left[1 - \exp\left(-\frac{b_2 t}{\beta_4}\right)\right] \quad . \tag{4.2a,b}$$

This Cosserat solution may be compared with the exact solution recorded in [4, p. 112] by defining the average temperature θ^*_{avg} and the average temperature gradient ϕ^*_{avg} in the direction normal to the plate's surface. It follows that

$$\theta^*_{avg} \equiv \theta_o + \frac{1}{h}\int_{-h/2}^{h/2} (\theta^* - \theta_o)d\theta^3 = \theta_o - \left(\frac{q^+}{\rho^*_o\ ch}\right)t \quad , \tag{4.3a}$$

$$\phi^*_{avg} \equiv \frac{12}{h^3}\int_{-h/2}^{h/2} (\theta^* - \theta_o)\theta^3\ d\theta^3 = -\left(\frac{q^+}{2K}\right)\left[1 - \frac{96}{\pi^4}\sum_{n=1}^{\infty}\frac{\exp\left\{-\frac{(2n-1)^2 K\pi^2 t}{\rho^*_o\ c\ h^2}\right\}}{(2n-1)^4}\right] , \tag{4.3b}$$

where K is the thermal conductivity and c is the specific heat at zero strain. For most practical purposes, we may retain only the first term in the series in (4.3b) so that the Cosserat solution and the exact solution have nearly the same form (note that $96/\pi^4 = 0.9855$). Equating these solutions, we obtain

$$b_2 = \frac{Kh}{\theta_o} \quad , \quad \beta_3 = \frac{\rho^*_o\ c\ h}{\theta_o} \quad , \quad \beta_4 = \frac{\rho^*_o\ c\ h^3}{\pi^2\ \theta_o} \quad . \tag{4.4a,b,c}$$

Next, we compare with exact steady state solutions for a rigid heat conductor of the forms $\theta^* = a\theta^1$ and $\theta^* = a\theta^1\theta^2$, where a is a constant. This comparison yields

$$a_o = \frac{Kh}{\theta_o} \quad , \quad b_1 = \frac{Kh^3}{12\ \theta_o} \quad . \tag{4.5a,b}$$

The last thermal coefficient β_5 corresponds to the arbitrary constant reference value of the entropy and therefore cannot be specified.

Apart from minor differences in sign, which are caused by our constitutive assumption (2.13a), all of the values for the thermal coefficients determined here are the same as those proposed in [2], except that for β_4. The value for β_4 proposed in [2],

$$\beta_4 = \frac{\rho_o^* h^3}{12\ \theta_o} \tag{4.6}$$

was obtained by the method of direct integration of the constitutive equations instead of by the method of comparison with exact solutions. Because we ultimately require the Cosserat theory to reproduce exact results with as little error as possible, we adopt the latter approach and specify β_4 by (4.4c) instead of (4.6).

5. Determination of the Mechanical Coefficients α_3 and α_8

The discussion of constitutive equations in [1] emphasized that in general, the constitutive coefficients are not constants. In other words, values that are obtained by comparing Cosserat solutions with exact solutions of one class of problems may not be the same as values obtained by considering another class of problems. For example, values that predict accurate results for certain quantities in static problems may be different from those that predict accurate results in dynamic problems. Nevertheless, from a practical point of view we need to specify values for the constitutive coefficients. By way of background, we recall [1] that the coefficients α_3 and α_8 associated with tangential shear deformation* have been specified by

*For a shell ($B_{\alpha\beta} \neq 0$) the coefficient α_8 relates $M^{3\alpha}$ in (2.15e) to tangential shear γ_α (see 2.61).

$$\alpha_3 = \frac{5}{6}\,\mu h \quad \text{or} \quad \alpha_3 = \frac{\pi^2}{12}\,\mu h \quad , \quad \alpha_8 = \frac{7}{120}\,\mu h^3 \quad . \tag{5.1a,b,c}$$

The values (5.1a,c) can be motivated by assuming a form for the stress distribution through the thickness of a plate and developing an approximate expression for the strain energy function by integration [1,5]. However, the value (5.1b) may be obtained by comparing with a solution for a vibrating plate [6]. Thus, we would expect that either of the values (5.1a,b) for α_3 (those values are very close to each other) would be appropriate for dynamical problems. However, as was pointed out in [1], the value (5.1c) for α_8 has not been validated by comparison with any exact solution. Here, we determine different values for α_3 and α_8 by comparing with exact solutions of two static problems.

First, we determine α_3 by considering the static isothermal problem of simple shear in the $\theta^1 - \theta^3$ plane of a plate in the absence of body force ($\underset{\sim}{f}^* = 0$). For this problem, we specify $\underset{\sim}{t}^+ = \tau\,\underset{\sim}{A}_1$, $\underset{\sim}{t}^- = -\tau\,\underset{\sim}{A}_1$, where τ is the shear stress applied to the major surfaces of the plate. It follows that the Cosserat solution is an exact solution if we specify

$$\alpha_3 = \mu h \quad . \tag{5.2}$$

A discussion of this result will be given at the end of this section.

Next, the coefficient α_8 will be determined by considering the static, isothermal problem of a circular cylindrical shell of radius R and thickness h with its inner surface held fixed and its outer surface rotated by an amount κh (Figure C.1). Let $\underset{\sim}{e}_i$ be an orthonormal coordinate system with $\underset{\sim}{e}_1$ parallel to the circumferential tangent to the shell and $\underset{\sim}{e}_3$ parallel to the outward normal to the shell (Figure C.1). Further, let r^* be the distance of a point from the symmetry axes of the shell, u_1^* be the displacement in the $\underset{\sim}{e}_1$ direction, and t_{13} be the (1,3) component of the Cauchy stress t_{ij}. Then the exact three-dimensional solution yields

$$r^* = R + \theta^3 \quad , \tag{5.3a}$$

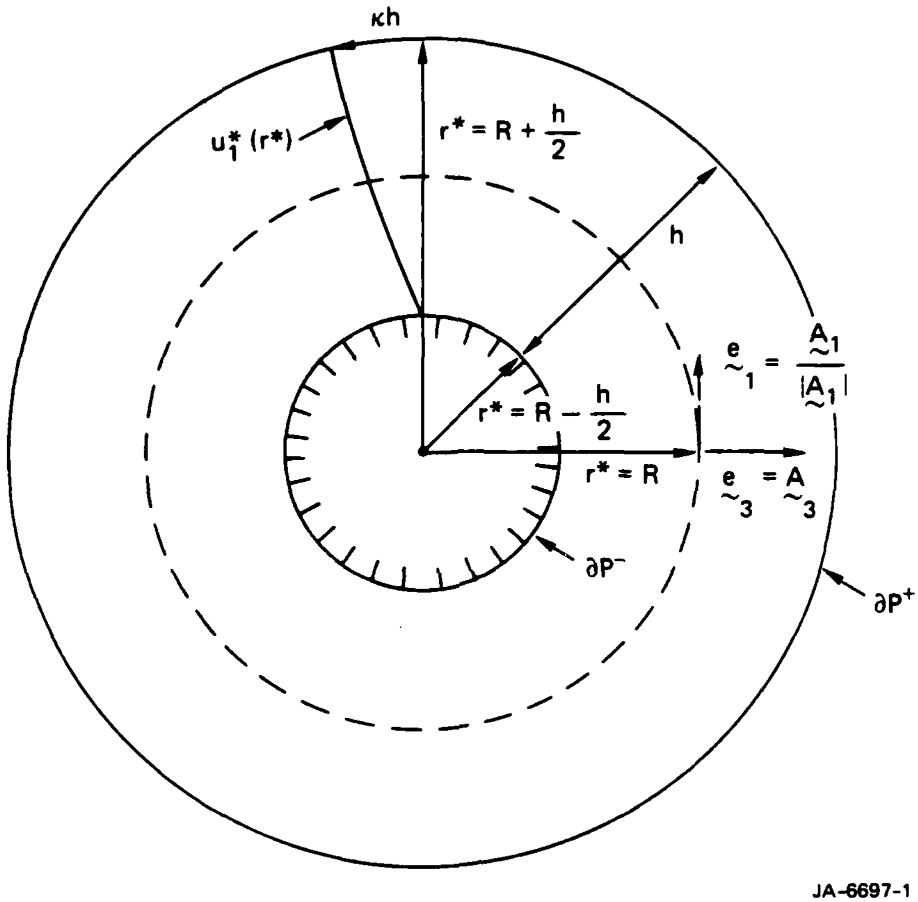

FIGURE C.1 CROSS SECTION OF A CIRCULAR CYLINDRICAL SHELL WITH THE INNER SURFACE ∂P^- HELD FIXED AND THE OUTER SURFACE ∂P^+ ROTATED BY AN AMOUNT κh

For this solution $R = h$.

$$u_1^*(r^*) = \frac{1}{2}(1 + \frac{h}{2R})\ \kappa\ [r^* - (1 - \frac{h}{2R})^2\ \frac{R^2}{r^*}]\quad , \tag{5.3b}$$

$$t_{13}(r^*) = t_{31}(r^*) = \mu(1 + \frac{h}{2R})(1 - \frac{h}{2R})^2\ \frac{R^2}{(r^*)^2}\ \kappa\quad , \tag{5.3c}$$

where all other components of the displacement u_i^* and stress t_{ij} vanish. Thus, the displacements $\underset{\sim}{u}^+$ and $\underset{\sim}{u}^-$, and traction vectors $\underset{\sim}{t}^+$ and $\underset{\sim}{t}^-$ on the top surface ∂P^+ and bottom surface ∂P^-, respectively, are given by

$$\underset{\sim}{u}^+ = u_1^*(R + \frac{h}{2})\underset{\sim}{e}_1 = \kappa h\ \underset{\sim}{e}_1\quad ,\quad \underset{\sim}{u}^- = u_1^*(R - \frac{h}{2})\underset{\sim}{e}_1 = 0\quad , \tag{5.4a,b}$$

$$\underset{\sim}{t}^+ = t_{13}(R + \frac{h}{2})\underset{\sim}{e}_1 = \mu\ \frac{(1 - \frac{h}{2R})^2}{(1 + \frac{h}{2R})}\ \kappa\ \underset{\sim}{e}_1 = t^+\ \underset{\sim}{e}_1\quad , \tag{5.4c}$$

$$\underset{\sim}{t}^- = - t_{13}(R - \frac{h}{2})\underset{\sim}{e}_1 = - \mu(1 + \frac{h}{2R})\kappa\ \underset{\sim}{e}_1 = - t^-\ \underset{\sim}{e}_1\quad , \tag{5.4d}$$

where t^+ and t^- are the values of the stress t_{13} on the top and bottom surfaces, respectively. The solution $u_1^*(r^*)$ is plotted in Figure C.1 for the thick-shell case where $R = h$. Notice that since u_1^* is nearly linear in r^*, we should expect the Cosserat solution to predict accurate results even in the limit of a thick shell.

Now, the geometry of a cylindrical shell of radius R is characterized by

$$\underset{\sim}{A}_1 = R\ \underset{\sim}{e}_1\quad ,\quad \underset{\sim}{A}_2 = \underset{\sim}{e}_2\quad ,\quad \underset{\sim}{A}_3 = \underset{\sim}{e}_3\quad ,\quad \Gamma^\sigma_{\alpha\beta} = \underset{\sim}{A}_{\alpha,\beta} \cdot \underset{\sim}{A}^\sigma = 0\quad , \tag{5.5a,b,c,d}$$

$$B_{11} = - R\quad ,\quad \text{all other } B_{\alpha\beta} = 0\quad , \tag{5.5e,f}$$

$$B^1_1 = - \frac{1}{R}\quad ,\quad \text{all other } B^\beta_\alpha = 0\quad , \tag{5.5g,h}$$

where $\Gamma^\sigma_{\alpha\beta}$ is the Christoffel symbol. Using (2.5), (2.17a) and (5.5), we realize that the conditions (5.4a,b) yield

$$\underset{\sim}{u}^{+} = \underset{\sim}{u} + \frac{h}{2}\underset{\sim}{\delta} = \kappa h\, \underset{\sim}{e}_1 \quad , \quad \underset{\sim}{u}^{-} = \underset{\sim}{u} - \frac{h}{2}\underset{\sim}{\delta} = 0 \tag{5.6a,b}$$

Consistent with (5.6) we may take

$$u^1 = \frac{h\kappa}{2R} \quad , \quad u^2 = u^3 = 0 \quad , \quad \delta^1 = \frac{\kappa}{R} \quad , \quad \delta^2 = \delta^3 = 0 \quad , \tag{5.7a,b,c,d}$$

so that

$$\gamma_1 = R\kappa\left(1 - \frac{h}{2R}\right) \quad , \quad \kappa_{31} = -\kappa\left(1 - \frac{h}{2R}\right) \quad , \tag{5.8a,b}$$

$$e_{\alpha\beta} = 0 \quad , \quad \gamma_2 = \gamma_3 = 0 \quad , \quad \kappa_{\alpha\beta} = 0 \quad , \quad \kappa_{32} = 0 \quad . \tag{5.8c,d,e,f}$$

It follows that the only nontrivial equations of equilibrium become

$$\rho_o\left(f^1 + \frac{1}{R}\ell^1\right) = 0 \quad , \quad 0 = \rho_o\,\ell^1 - k^1 + \frac{1}{R}M^{31} \tag{5.9a,b}$$

where

$$\rho_o\, f^1 = \frac{1}{R}\left[t^{+}\left(1 + \frac{h}{2R}\right) - t^{-}\left(1 - \frac{h}{2R}\right)\right] \quad , \tag{5.10a}$$

$$\rho_o\, \ell^1 = \frac{1}{R}\left[t^{+}\left(\frac{h}{2}\right)\left(1 + \frac{h}{2R}\right) + t^{-}\left(\frac{h}{2}\right)\left(1 - \frac{h}{2R}\right)\right] \quad , \tag{5.10b}$$

$$k^1 = \left(\frac{\alpha_3\,\kappa}{R}\right)\left(1 - \frac{h}{2R}\right) \quad , \quad M^{31} = -\left(\frac{\alpha_8\,\kappa}{R^2}\right)\left(1 - \frac{h}{2R}\right) \quad . \tag{5.10c,d}$$

Substituting (5.10) into (5.9) and comparing the result with (5.4), we realize that the Cosserat equations will predict exact values for t^+ and t^- if α_3 and α_8 satisfy the equation

$$\alpha_3 + \frac{\alpha_8}{R^2} = \mu h \quad . \tag{5.11}$$

Now if we adopt the specification (5.2), we deduce that

$$\alpha_8 = 0 \quad . \tag{5.12}$$

We are now in a position to discuss the significance of the results (5.2) and (5.12). Since the value of α_3 given by (5.2) was obtained by comparing with the exact solution of the static simple shear problem, it follows that if α_3 were specified by (5.1a,b) instead of (5.2), the Cosserat equations would necessarily predict incorrect results for simple shear which would be undesirable. However, one could question whether it is of prime importance for a shell theory to predict results of simple shear--which requires surface tractions to be specified on the major surfaces--when many applications of shell theory consider shells with free major surfaces. In response to this question, we note that two important problems of a plate with free major surfaces have been solved for arbitrary values of α_3. Problem A [1, Sec. 24] considers pure twist of a plate and problem B [7] considers pure bending of a plate with a circular hole. If the value [5.2] for α_3 is adopted instead of (5.1a), then it can be shown that the solution of problem A is slightly improved and the solution of problem B is only slightly modified with η being replaced by $(6/5)^{1/2}\ \eta = 1.10\ \eta$ in the formulas in [7].

It is also of interest to note that if we were to specify alternative constitutive equations by replacing $\kappa_{1\alpha}$ in (2.13a) with $\rho_{1\alpha}$, then for the example associated with Figure C.1, the quantity M^{31} would vanish and equations (5.9) would again yield the result (5.2). With reference to this same example, we observe that there is an inconsistency between the specifications (5.1) and the result (5.11). In particular, it is clear that if either (5.1a) or (5.1b) are substituted into (5.11), we conclude that α_8 is proportional to $\mu h R^2$, which is significantly different than (5.1c).

In conclusion; it appears that for static problems it is better to specify α_3 by (5.2) instead of (5.1a,b) and α_8 by (5.12) instead of (5.1c). This is because with this specification, the Cosserat theory predicts accurate results for all four static problems considered in this section without the inconsistency described above. However, if comparison with the dynamic problem considered in [6] is of prime

importance, then it is best to specify α_3 by (5.1b) instead of (5.2). In conclusion, we recall the observation in [1] that the constitutive coefficients for shell theory are not constants in the sense that the best values for these coefficients may be problem dependent.

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Appendix D

HEAT CONDUCTION IN PLATES AND SHELLS WITH
EMPHASIS ON A CONICAL SHELL

(Submitted for publication in the International
Journal of Solids and Structures)

ABSTRACT

This paper is concerned with analyzing heat conduction in rigid shell-like bodies. The thermal equations of the theory of a Cosserat surface are used to calculate the average (through-the-thickness) temperature and temperature gradient directly. Specific attention is focused on a conical shell. The conical shell is particularly interesting because it has a converging geometry such that the shell near its tip is "thick" even though the shell near its base may be "thin." Generalized constitutive equations, which include certain geometrical features of shells, are developed here in a consistent manner. These equations are tested by considering a number of problems of plates, circular cylindrical shells, and spherical shells, and comparing the results with exact solutions. In all cases, satisfactory results are predicted even in the thick-shell limit. Finally, a problem of transient heat conduction in a conical shell is solved. It is shown that the thermal bending moment produced by the average temperature gradient is quite severe near the tip and it attains its maximum value in a relatively short time.

1. Introduction

Most aerospace structures are compositions of structural components that can be modeled as shell-like bodies. For various reasons, it is desirable to determine the thermomechanical response of these shell-like bodies to thermal and mechanical loads. Within the context of classical linear shell theory, the temperature distribution influences the mechanical response of the shell through the resultant thermal force and resultant thermal moment. For an elastic shell, the thermal force is related to the average (through-the-thickness) temperature and the thermal moment is related to the average temperature gradient by constitutive equations.

If the strain rate in a given problem is sufficiently small, then the thermal and mechanical problems uncouple in the sense that the temperature field may be determined by solving equations for a rigid heat conductor. Then the resulting temperature field may be used to calculate the thermal force and thermal moment, which provide thermal loading for determining shell deformation.

In this paper, we confine attention to determining the temperature distribution in a shell-like body that is treated as a rigid heat conductor. Although the temperature distribution can be determined by attempting to solve the three-dimensional heat conduction equation, this approach has two major disadvantages. First, because the thermal loads for shell theory depend only on the average temperature and temperature gradient, much of the details calculated by this approach are not of prime importance. Second, because the heat conduction equation admits separable solutions for only a limited number of geometries, it is exceedingly difficult to obtain analytical solutions for many typical shell geometries. This latter problem has been addressed in [1], where equations are developed to calculate an approximate temperature distribution in shells of revolution.

We take a different approach and use thermal equations for shells that have recently been developed [2] to predict the average temperature and temperature gradient directly. These equations are based on

modeling the shell-like body as a Cosserat surface. Details of this theory may be found in [2,3]. Specifically, the objective of this paper is to determine the average temperature and temperature gradient in a conical shell (Figure D.1), which is a basic aerospace structure. The conical shell is particularly interesting because it has a converging geometry so tha the shell near its tip is necessarily "thick" even though the shell near its base may be "thin." For this reason, it is questionable whether any shell theory can accurately predict results for the critical tip region. Here it is shown that with appropriate constitutive equations, the Cosserat theory includes enough of the geometry of the shell to predict relatively accurate results for the conical shell.

It is not a trivial matter to develop equations for shells that produce reasonable results in the thick-shell limit. For example, we recall that the equations in [1] were developed by writing the heat conduction equation in a form appropriate for shells and then neglecting quantities multiplied by higher powers of the ratio of the thickness to radius of curvature. Even though these equations are more complicated than the Cosserat equations in that details of the through-the-thickness temperature distribution are calculated, too much of the shell geometry has been neglected, and hence they predict inaccurate results in the thick-shell limit. The predictions of the equations in [1] are compared with the more accurate predictions of the Cosserat theory for the thick-shell problems considered in Sections 4 and 5.

In the following sections, we discuss the basic equations of the Cosserat theory and then solve a number of problems. To develop confidence in the predictions of the Cosserat theory in the base region of the conical shell, we solve various problems for a plate and compare with exact solutions in [4]. These problems examine the effect of the three types of boundary conditions on the major surfaces of the plate: specified heat flux, specified temperature, and radiation. Next, to develop confidence in the predictions of the theory in the tip region of the conical shell, we use the same equations to solve specific problems

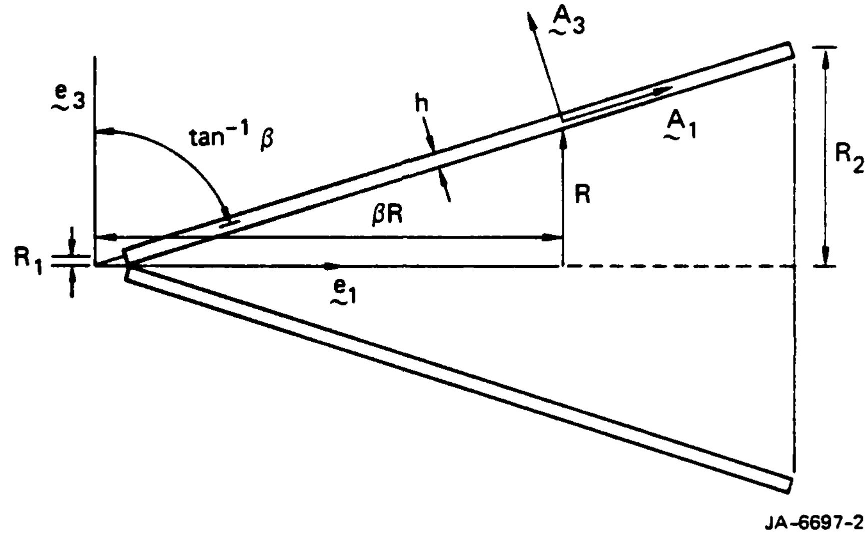

FIGURE D.1 CONICAL SHELL OF CONSTANT THICKNESS h, TIP RADIUS R_1, AND BASE RADIUS R_2

for a solid circular cylinder and a solid sphere, and compare the results with exact solutions. Finally, after having developed confidence in the predictions of the theory in both the tip and base regions of the conical shell, we solve a specific heat conduction problem for a conical shell.

2. Basic Equations

Let the material points of the Cosserat surface C be identified by means of a system of convected coordinates $\theta^\alpha (\alpha = 1,2)$ and let the two-dimensional region of space occupied by the material surface in the present configuration at time t be denoted by c. Further, let the vector valued function $\underset{\sim}{r}$ define the position of a material point of the surface C and at each such point define the vector valued function $\underset{\sim}{d}$, called the director, and the two temperature fields θ and ϕ, each referred to the present configuration. Then a thermomechanical process of the Cosserat surface is defined by

$$\underset{\sim}{r} = \underset{\sim}{r}(\theta^\alpha, t) \quad , \quad \underset{\sim}{d} = \underset{\sim}{d}(\theta^\alpha, t) \quad , \quad [\underset{\sim}{a}_1 \; \underset{\sim}{a}_2 \; \underset{\sim}{d}] > 0 \quad , \qquad (2.1a,b,c)$$

$$\theta = \theta(\theta^\alpha, t) \quad , \quad (\theta > 0) \quad , \quad \phi = \phi(\theta^\alpha, t) \quad , \qquad (2.1d,e,f)$$

where the tangent vectors $\underset{\sim}{a}_\alpha$ and the unit normal vector $\underset{\sim}{a}_3$ are defined by

$$\underset{\sim}{a}_\alpha = \frac{\partial \underset{\sim}{r}}{\partial \theta^\alpha} \; , \; \underset{\sim}{a}_\alpha \cdot \underset{\sim}{a}_3 = 0 \quad , \quad \underset{\sim}{a}_3 \cdot \underset{\sim}{a}_3 = 1 \quad , \quad a^{1/2} = [\underset{\sim}{a}_1 \; \underset{\sim}{a}_2 \; \underset{\sim}{a}_3] > 0 \quad , \qquad (2.2a,b,c,d)$$

and the condition (2.1c) ensures that the director is nowhere tangent to c. Also, in the above, θ represents the average (through-the-thickness) temperature in the shell and ϕ represents the average temperature gradient.

In the reference configuration, we assume that the shell has uniform thickness h and is at uniform temperature θ_0. Then, the

reference values of the various kinematic quantities may be denoted by*

$$\underset{\sim}{r} = \underset{\sim}{R} \ , \quad \underset{\sim}{d} = \underset{\sim}{D} = \underset{\sim}{A}_3 \ , \quad \underset{\sim}{a}_i = \underset{\sim}{A}_i \ , \quad a^{1/2} = A^{1/2} \ , \qquad (2.3a,b,c,d)$$

$$\theta = \theta_o \ , \quad \phi = 0 \ , \qquad (2.3e,f)$$

where $\underset{\sim}{R}$, $\underset{\sim}{A}_i$ and $A^{1/2}$ depend on the coordinate θ^α only. For the rigid heat conductor considered here, there is no distinction between the reference and the present configurations; hence equations (2.3a-d) hold for all time. Further, all tensor quantities will be referred to the base vectors $\underset{\sim}{A}_i$ and their reciprocals $\underset{\sim}{A}^i$ defined by

$$\underset{\sim}{A}_i \cdot \underset{\sim}{A}^j = \delta_i^{\ j} \ , \qquad (2.4)$$

where $\delta_i^{\ j}$ is the Kronecker symbol.

Let P, bounded by the closed curve ∂P, denote the region occupied by an arbitrary material portion of the surface c and let $\underset{\sim}{\nu}$ be the unit outward normal to ∂P. Using the notation of [2], we define the following quantities: the positive mass density (mass per unit area of P) in the reference configuration $\rho_o = \rho_o(\theta^\alpha)$; the specific (per unit mass of P) entropies $\eta = \eta(\theta^\alpha,t)$ and $\eta_1 = \eta_1(\theta^\alpha,t)$; the specific internal rates of production of entropy $\xi = \xi(\theta^\alpha,t)$, $\xi_1 = \xi_1(\theta^\alpha,t)$, and $\bar{\xi}_1 = \bar{\xi}_1(\theta^\alpha,t)$; the entropy fluxes $k = k(\theta^\alpha,t;\ \underset{\sim}{\nu})$ and $k_1 = k_1(\theta^\alpha,t;\ \underset{\sim}{\nu})$, each per unit length of the curve ∂P; the specific external rates of supply of entropy $s = s(\theta^\alpha,t)$ and $s_1 = s_1(\theta^\alpha,t)$; the specific internal energy $\varepsilon = \varepsilon(\theta^\alpha,t)$; and the specific Helmholtz free energy $\psi = \psi(\theta^\alpha,t) \equiv$
$\varepsilon - \theta\eta - \phi\eta_1$. With suitable continuity assumptions, it can be shown that [2,3]

$$k = \underset{\sim}{p} \cdot \underset{\sim}{\nu} = p^\alpha \nu_\alpha \ , \quad k_1 = \underset{\sim}{p}_1 \cdot \underset{\sim}{\nu} = p_1^\alpha \nu_\alpha \ , \qquad (2.5a,b)$$

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*Throughout the text Greek indices have a range (1,2) and Latin indices have a range (1,2,3).

where $\nu_\alpha = \underset{\sim}{A}_\alpha \cdot \underset{\sim}{\nu}$ are the components of the normal vector $\underset{\sim}{\nu}$ and where we use the usual summation convention over repeated indices. Further, with reference to the energy equation, the specific external rates of heat supply r and r_1; and the heat flux vectors $\underset{\sim}{q}$ and $\underset{\sim}{q}_1$ are defined by

$$r = \theta s \quad , \quad r_1 = \phi s_1 \quad , \quad \underset{\sim}{q} = \theta \underset{\sim}{p} \quad , \quad \underset{\sim}{q}_1 = \phi \underset{\sim}{p} \quad . \qquad (2.6a,b,c,d)$$

Now, the local forms of the balances of entropy may be recorded as [2]

$$\rho_o \dot{\eta} = \rho_o(s + \xi) - p^\alpha|_\alpha \quad , \quad \rho_o \dot{\eta}_1 = \rho_o(s_1 + \xi_1) - p_1{}^\alpha|_\alpha \quad , \qquad (2.7a,b)$$

where a dot denotes material time differentiation and where a bar denotes covariant differentiation with respect to the metric $A_{\alpha\beta}$ of the shell surface. For later convenience, we recall [2,3] definitions for the metric tensor $A_{\alpha\beta}$, and its reciprocal $A^{\alpha\beta}$, the curvature tensor $B_{\alpha\beta}$, the Christoffel symbol $\Gamma^\sigma_{\alpha\beta}$, and covariant differentiation in the forms

$$A_{\alpha\beta} = \underset{\sim}{A}_\alpha \cdot \underset{\sim}{A}_\beta \quad , \quad A^{\alpha\beta} = \underset{\sim}{A}^\alpha \cdot \underset{\sim}{A}^\beta \quad , \quad B_{\alpha\beta} = A_{\alpha,\beta} \cdot \underset{\sim}{A}_3 \quad , \qquad (2.8a,b,c)$$

$$\Gamma^\sigma_{\alpha\beta} = \underset{\sim}{A}_{\alpha,\beta} \cdot \underset{\sim}{A}^\sigma \quad , \quad \theta_{|\alpha} = \theta_{,\alpha} \quad , \quad p^\sigma|_\beta = p^\sigma{}_{,\beta} - \Gamma^\sigma_{\alpha\beta}\, p^\alpha \quad , \qquad (2.8d,e,f)$$

where a comma denotes partial differentiation with respect to θ^α.

Equations (2.7) must be supplemented by an energy equation and constitutive equations. It follows from [2] that the energy equation for a rigid thermoelastic shell is satisfied provided that

$$\eta = -\frac{\partial\psi}{\partial\theta} \quad , \quad \eta_1 = -\frac{\partial\psi}{\partial\phi} \quad , \qquad (2.9a,b)$$

and

$$\rho_o\, \theta\, \xi + \rho_o\, \phi\, \xi_1 + \underset{\sim}{p} \cdot \underset{\sim}{g} + \underset{\sim}{p}_1 \cdot \underset{\sim}{g}_1 = 0 \qquad (2.10)$$

where the temperature gradients $\underset{\sim}{g}$ and $\underset{\sim}{g}_1$ are defined by

$$\underset{\sim}{g} = \theta_{,\alpha} \underset{\sim}{A}^{\alpha} \quad , \quad \underset{\sim}{g}_1 = \phi_{,\alpha} \underset{\sim}{A}^{\alpha} \quad . \tag{2.11a,b}$$

Confining attention to a rigid shell that is thermally isotropic, we specify constitutive equations in the form

$$2 \rho_o \psi = - \beta_3(\theta^2 - 2 \theta \theta_o) - \beta_4 \phi^2 - 2 \beta_5 \theta \quad , \tag{2.12a}$$

$$\underset{\sim}{p} = - a_o \underset{\sim}{g} \quad , \quad \underset{\sim}{p}_1 = - b_1 \underset{\sim}{g}_1 \quad , \tag{2.12b,c}$$

$$\rho_o \theta \xi = a_o \underset{\sim}{g} \cdot \underset{\sim}{g} + b_1 \underset{\sim}{g}_1 \cdot \underset{\sim}{g}_1 + b_2 \phi^2 \quad , \tag{2.12d}$$

$$\rho_o \xi_1 = \rho_o \bar{\xi}_1 = - b_2 \phi \tag{2.12e}$$

where a_o, b_1, b_2, $\beta_3 - \beta_5$ are constants. Substituting (2.12a) into (2.9), we have

$$\rho_o \eta = \beta_3(\theta - \theta_o) + \beta_5 \quad , \quad \rho_o \eta_1 = \beta_4 \phi \quad . \tag{2.13a,b}$$

The form of the constitutive equations (2.12) represents a slight generalization of those introduced in [2] for the linear theory.* These equations are chosen to automatically satisfy the reduced energy equation (2.10) without approximation.

In postulating the form of the constitutive equations (2.12), we tacitly assume that constitutive equations that are valid for a plate are also valid for a shell. In the discussion in Section 5, we observe that certain geometrical features of the shell must be included in the constitutive equations to predict relatively accurate results for a solid sphere. These geometrical features of the shell may be introduced

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*The sign of the constants β_3, β_4 used here is opposite that used in [2].

by appropriately modifying the constitutive equations to take the forms

$$2\,\rho_o^*\,h\,\phi = -\,\beta_3(\theta^2 - 2\,\theta\,\theta_o) - \beta_4\,\phi^2 - 2\,\beta_5\,\theta \quad , \tag{2.14a}$$

$$\underset{\sim}{p} = -\,a_o\,\underset{\sim}{g} \quad , \quad \underset{\sim}{p}_1 = -\,b_1\,\underset{\sim}{g}_1 \quad , \tag{2.14b,c}$$

$$\rho_o\,\theta\,\xi = a_o\,\underset{\sim}{g}\cdot\underset{\sim}{g} + b_1\,\underset{\sim}{g}_1\cdot\underset{\sim}{g}_1 + \left(\frac{\rho_o}{\rho_o^*\,h}\right) b_2\,\phi^2 \quad , \tag{2.14d}$$

$$\rho_o^*\,h\,\xi_1 = \rho_o^*\,h\,\bar{\xi}_1 = -\,b_2\,\phi \quad , \tag{2.14e}$$

$$\rho_o\,\eta = \frac{\rho_o}{\rho_o^*\,h}\,[\beta_3(\theta - \theta_o) + \beta_5] \quad , \quad \rho_o\,\eta_1 = \frac{\rho_o}{\rho_o^*\,h}\,\beta_4\,\phi \quad , \tag{2.14f,g}$$

where ρ_o^* is the constant three-dimensional mass density (mass per unit volume) of the material and h is the constant thickness of the shell. The constitutive equations (2.14f,g) depend on the geometry of the shell through the ratio $\rho_o/\rho_o^*\,h$ [see equation (2.16a)].

Within the context of the general theory, the constitutive equations must be further restricted by statements of the second law of thermodynamics [2]. For either of the sets of constitutive equations (2.12) and (2.13) or (2.14), these restrictions reduce to

$$a_o \geq 0 \quad , \quad b_1 \geq 0 \quad , \quad b_2 \geq 0 \quad , \quad \beta_3 > 0 \quad . \tag{2.15a,b,c,d}$$

To linearize the equations presented above, we assume that the temperatures $(\theta - \theta_o)$ and ϕ, and their space and time derivatives are small enough that quadratic expressions in these quantities may be neglected relative to linear expressions. It follows from (2.12d) or (2.14d) that ξ is of higher order so that ξ may be set equal to zero in (2.7a).

Now, we recall [2,3] that the Cosserat theory developed by direct approach may be brought into a one-to-one correspondence with the three-dimensional theory by assuming that the position vector $\underset{\sim}{r}^*$ of a point in

the shell and the temperature field θ^* admit the representations

$$\underset{\sim}{r}^* = \underset{\sim}{r}^*(\theta^\alpha,\theta^3,t) = \underset{\sim}{r}(\theta^\alpha,t) + \theta^3\ \underset{\sim}{d}(\theta^\alpha,t) \quad , \tag{2.15a}$$

$$\theta^* = \theta^*(\theta^\alpha,\theta^3,t) = \theta(\theta^\alpha,t) + \theta^3\ \phi(\theta^\alpha,t) \quad , \tag{2.15b}$$

where θ^3 is a coordinate through the thickness of the shell. For a shell of constant thickness h, we may choose the reference surface of the shell to be the middle surface and define the top surface ∂P^+ of the shell by $\theta^3 = h/2$ and the bottom surface ∂P^- by $\theta^3 = -h/2$. If the three-dimensional mass density ρ_o^* of the shell is constant, then it may be shown that [2,3,5]:

$$\lambda = \rho_o\ A^{1/2} = \int_{-h/2}^{h/2} \rho_o^*\ G^{1/2}\ d\theta^3 = (\rho_o^*\ h\ A^{1/2})[1 + \frac{h^2}{12}(B_1^1\ B_2^2 - B_1^2\ B_2^1)] \quad , \tag{2.16a}$$

$$\lambda s = \lambda\hat{s} - B^+\ k^+ - B^-\ k^- \quad , \quad \lambda\hat{s} = \int_{-h/2}^{h/2} \rho_o^*\ s^*\ G^{1/2}\ d\theta^3 \quad , \tag{2.16b,c}$$

$$\lambda s_1 = \lambda\hat{s}_1 - (\frac{h}{2})\ B^+\ k^+ + (\frac{h}{2})\ B^-\ k^- \quad , \quad \lambda\hat{s}_1 = \int_{-h/2}^{h/2} \rho_o^*\ s^*\ G^{1/2}\ \theta^3\ d\theta^3 \quad , \tag{2.16d,e}$$

$$G^{1/2} = A^{1/2}[1 - \theta^3\ B_\sigma^\sigma + (\theta^3)^2\ (B_1^1\ B_2^2 - B_1^2\ B_2^1)] \quad , \tag{2.16f}$$

$$B^+ = A^{1/2}[1 - \frac{h}{2}\ B_\sigma^\sigma + \frac{h^2}{4}\ (B_1^1\ B_2^2 - B_1^2\ B_2^1)] \quad , \tag{2.16g}$$

$$B^- = A^{1/2}[1 + \frac{h}{2}\ B_\sigma^\sigma + \frac{h^2}{4}\ (B_1^1\ B_2^2 - B_1^2\ B_1^2)] \quad , \tag{2.16h}$$

$$\theta^+ = \theta + \frac{h}{2}\ \phi \quad , \quad \theta^- = \theta - \frac{h}{2}\ \phi \quad , \tag{2.16i,j}$$

where s^* is the three-dimensional rate of entropy supply; k^+ and k^- are, respectively, the entropy fluxes applied to the major surfaces ∂P^+ and ∂P^-; B^α_β are the mixed components of the curvature tensor; and θ^+ and θ^- are, respectively, the temperatures on the major surfaces ∂P^+ and ∂P^-. Also, we note that for the linear theory

$$\theta_o\, k^+ = q^+ \quad , \quad \theta_o\, k^- = -\, q^- \quad , \tag{2.17a,b}$$

where q^+ is the heat flux measured positive for heat flowing out of the surface ∂P^+ and q^- is the heat flux measured positive for heat flowing into the surface ∂P^-.

Most of the constitutive coefficients were evaluated in [2] by direct integration of the three-dimensional constitutive equations. An alternative approach was taken in [5], where the coefficients were evaluated by comparing Cosserat solutions with exact three-dimensional solutions. Except for the value of β_3, the results in [2] and [5] are the same. Here, we adopt the results in [5] and specify

$$a_o = \frac{Kh}{\theta_o} \quad , \quad b_1 = \frac{Kh^3}{12\,\theta_o} \quad , \quad b_2 = \frac{Kh}{\theta_o} \quad , \tag{2.18a,b,c}$$

$$\beta_3 = \frac{\rho_o^* \, ch}{\theta_o} \quad , \quad \beta_4 = \frac{\rho_o^* \, ch^3}{\pi^2 \, \theta_o} \quad , \tag{2.18d,e}$$

where K is the thermal conductivity and c is the specific heat at constant strain of the material. The coefficient β_5 corresponds to the arbitrary constant reference value of the entropy and therefore cannot be specified. Since the material constants K and c are positive, we realize from (2.18) that the restrictions (2.15) are satisfied.

Finally, we use (2.14), (2.16)-(2.18) to write the linearized version of equations (2.7) in the form

$$\rho_o\, c\, \dot{\theta} = \rho_o\, \theta_o\, \hat{s} - A^{-1/2}\, [B^+ q^+ - B^- q^-] + Kh\, \nabla^2 \theta \quad , \tag{2.19a}$$

$$\frac{\rho_o\, ch^2}{\pi^2}\,\dot{\phi} = \rho_o\, \theta_o\, \hat{s}_1 - A^{-1/2}\left(\frac{h}{2}\right)[B^+ q^+ + B^- q^-]$$

$$- \frac{\rho_o}{\rho_o^* h}\, Kh\, \phi + \frac{Kh^3}{12}\nabla^2\phi \quad , \qquad (2.19b)$$

where the Laplacian operator $\nabla^2\,\theta$ is defined by

$$\nabla^2\,\theta = A^{\alpha\beta}\,\theta_{|\alpha\beta} = A^{\alpha\beta}(\theta_{,\alpha\beta} - \Gamma^{\sigma}_{\alpha\beta}\,\theta_{,\sigma}) \quad . \qquad (2.20)$$

3. Plates

In this section, we examine the validity of the Cosserat theory in the thin-shell limit by considering three problems of heat conduction in a plate. These problems are chosen to examine the effects of specifying heat flux, temperature, or radiation-type boundary conditions on the major surfaces of the plate. For each of these problems, we neglect external entropy supply (or external heat supply) and consider temperature fields that are functions of time only so that

$$\hat{s} = 0 \quad , \quad \hat{s}_1 = 0 \quad , \quad \theta = \theta(t) \quad , \quad \phi = \phi(t) \quad . \qquad (3.1a,b,c,d)$$

Further, the curvature tensor $B_{\alpha\beta}$ for a plate vanishes. Hence, from (2.16) we deduce that

$$B_{\alpha\beta} = 0 \quad , \quad \rho_o = \rho_o^*\, h \quad , \quad B^+ = A^{1/2} \quad , \quad B^- = A^{1/2} \quad . \qquad (3.2a,b,c,d)$$

and that equations (2.19) reduce to

$$\rho_o^*\, ch\, \dot{\theta} = - q^+ + q^- \quad , \qquad (3.3a)$$

$$\frac{\rho_o^*\, ch^2}{\pi^2}\,\dot{\phi} = - \frac{1}{2}\, q^+ - \frac{1}{2}\, q^- - K\phi \quad . \qquad (3.3b)$$

Problem 1: For this problem, the heat flux q^+ is specified to be constant on the top surface, the bottom surface is insulated, and the plate is initially at uniform temperature θ_0. Mathematically, these conditions are characterized by

$$q^+ = \text{constant} \quad , \quad q^- = 0 \quad , \tag{3.4a,b}$$

$$\theta = \theta_0 \quad , \quad \phi = 0 \quad \text{at} \quad t = 0 \quad . \tag{3.4c,d}$$

Since the solution of equations (3.3) with the conditions (3.4) was developed in [5], we merely record the solution in the nondimensional form

$$\frac{-K(\theta - \theta_0)}{h\,q^+} = \tau \quad , \quad \frac{-K\phi}{q^+} = \frac{1}{2}\,[1 - \exp(-\pi^2\,\tau)] \quad , \tag{3.5a,b}$$

where τ is the nondimensional time parameter defined by

$$\tau = \frac{Kt}{\rho_0^* \, ch^2} \quad . \tag{3.6}$$

Recall from [5] that the constitutive coefficients were chosen by requiring the Cosserat solution to compare very well with the exact solution recorded in [4, p. 112].

To exhibit this comparison graphically, we have used (2.15b) to plot in Figure D.2 the Cosserat solution (3.5) together with the exact solution for various values of the time parameter τ. The dashed lines in Figure D.2 have been taken directly from [4, Figure 15, p. 113] and

$$x = \theta^3 + \frac{h}{2} \tag{3.7}$$

so that $x = 0$ locates the bottom surface ∂P^- and $x = h$ locates the top surface ∂P^+.

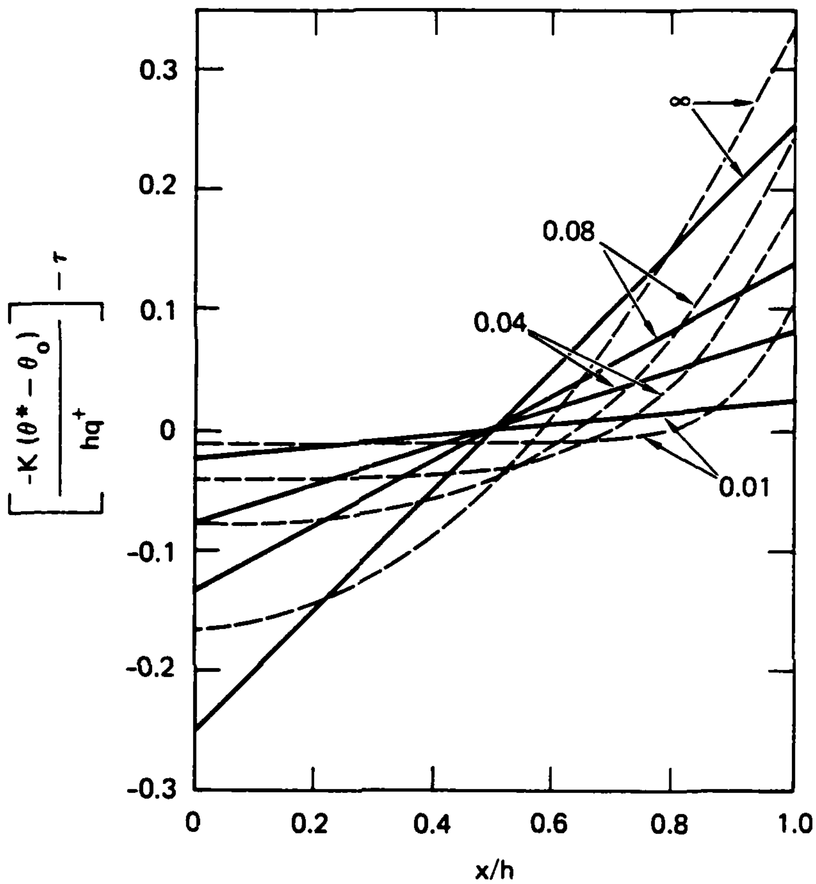

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FIGURE D.2 NORMALIZED TEMPERATURE IN A PLATE OF THICKNESS h, WITH ZERO HEAT FLUX AT x = 0, CONSTANT HEAT FLUX q^+ (OUT OF THE PLATE) AT x = h, AND UNIFORM INITIAL TEMPERATURE $\theta = \theta_o$

The numbers on the curves are values of $\tau = Kt/\rho_o^* ch^2$. The dashed lines are the exact solution and the solid lines are the Cosserat solution.

Problem 2: For this problem, the temperature θ^+ is specified as θ_o on the top surface, the heat flux q^- is specified to be constant on the bottom surface, and the plate is initially at uniform temperature θ_o. Mathematically, these conditions are characterized by

$$\theta^+ = \theta_o \quad , \quad q^- = constant \quad , \tag{3.8a,b}$$

$$\theta = \theta_o \quad , \quad \text{and} \quad \phi = 0 \quad \text{at} \quad t = 0 \quad . \tag{3.8c,d}$$

With the help of (2.16i) condition (3.8a) yields

$$\theta = \theta_o - \frac{h}{2}\phi \quad . \tag{3.9}$$

It is important to observe here that by specifying θ^+, we tacitly specify θ in terms of ϕ through equation (2.16i). It follows that it is not possible to specify independent initial values for θ and ϕ such as (3.8c,d). In other words, when temperature is specified on one or both of the major surfaces, we must, in general, modify the initial conditions. However, in the special case of this problem, conditions (3.8c,d) are consistent with (3.9).

Since θ^+ is specified, the heat flux q^+ must be determined from equations (3.3). Thus, using (3.9) in (3.3a) we deduce that

$$q^+ = q^- + \frac{\rho_o^* ch^2}{2}\dot{\phi} \quad . \tag{3.10}$$

Substituting (3.10) into (3.3b), we have

$$\rho_o^* ch^2 \left(\frac{4 + \pi^2}{4\pi^2}\right)\dot{\phi} + K\phi = -q^- \quad . \tag{3.11}$$

Now, solving (3.11) subject to the initial condition (3.8d), we may write the normalized solution in the form

$$\frac{K(\theta - \theta_o)}{hq^-} = \frac{1}{2}\left[1 - \exp\left\{-\left(\frac{4\pi^2}{4+\pi^2}\right)\tau\right\}\right] \quad , \tag{3.12a}$$

$$\frac{-K\phi}{q^-} = \left[1 - \exp\left\{-\left(\frac{4\pi^2}{4+\pi^2}\right)\tau\right\}\right] \quad , \tag{3.12b}$$

$$\frac{q^+}{q^-} = \left[1 - \left(\frac{2\pi^2}{4+\pi^2}\right)\exp\left\{-\left(\frac{4\pi^2}{4+\pi^2}\right)\tau\right\}\right] \quad . \tag{3.12c}$$

where τ is defined by (3.6).

To compare the Cosserat solution (3.12) with the exact solution recorded in [4, p. 113], we rewrite the exact solution in the form

$$\frac{K(\theta^* - \theta_o)}{hq^-} = \left(\frac{1}{2} - \theta^3\right) - \frac{8}{\pi^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}\sin\left[\left(\frac{2n+1}{2}\right)\pi\left(\frac{1}{2} - \theta^3\right)\right]$$

$$\times \exp\left[-\frac{(2n+1)^2\pi^2}{4}\tau\right] \quad . \tag{3.13}$$

Let us define the average temperature θ^*_{avg} and average temperature gradient ϕ^*_{avg} in the plate by the equations

$$\theta^*_{avg} - \theta_o = \frac{1}{h}\int_{-h/2}^{h/2}(\theta^* - \theta_o)\, d\theta^3 \quad , \tag{3.14a}$$

$$\phi^*_{avg} = \frac{12}{h^3}\int_{-h/2}^{h/2}(\theta^* - \theta_o)\,\theta^3\, d\theta^3 \quad . \tag{3.14b}$$

Then, substituting (3.13) into (3.14) and performing the integration, we deduce the results

$$\frac{K(\theta^*_{avg} - \theta_o)}{hq^-} = \frac{1}{2}\left[1 - \frac{32}{\pi^3}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^3}\exp\left\{-\frac{(2n+1)^2\pi^2}{4}\tau\right\}\right] \quad , \tag{3.15a}$$

$$\frac{-K\phi^*_{avg}}{q^-} = 1 - \frac{96}{\pi^4}\sum_{n=0}^{\infty}\left[\frac{4-(2n+1)\pi(-1)^n}{(2n+1)^4}\right]\exp\left[-\frac{(2n+1)^2\pi^2}{4}\tau\right] . \tag{3.15b}$$

Because the quantities θ and ϕ in the Cosserat solution correspond to θ^*_{avg} and ϕ^*_{avg}, we have plotted each of these in Figure D.3. The solid lines correspond to normalized values of θ and ϕ and the dashed lines correspond to normalized values of θ^*_{avg} and ϕ^*_{avg}. The comparison for all values of τ seems quite acceptable.

Problem 3: For this problem, we consider a plate of thickness 2h. The heat flux is specified appropriately for radiation from both the top and bottom surfaces and the plate is initially at a uniform temperature $\theta_o + V$. Mathematically, these conditions are characterized by

$$q^+ = KH(\theta^+ - \theta_o) \quad , \quad q^- = -KH(\theta^- - \theta_o) \quad , \tag{3.16a,b}$$

$$\theta = \theta_o + V \quad , \quad \text{and} \quad \phi = 0 \quad \text{at} \quad t = 0 \quad , \tag{3.16c,d}$$

where H is a constant specifying thermal radiation from the major surfaces. First, we will solve the problem as it is formulated in (3.16) and second we will obtain a more accurate solution by exploiting the symmetry about the center plane.

For the first solution, we substitute (2.16i,j) and (3.16a,b) into equations (3.3) and then replace h by 2h to obtain

$$\rho^*_o\, ch\, \dot{\theta} = -KH(\theta - \theta_o) \quad , \quad \frac{4\rho^*_o ch^2}{\pi^2}\dot{\phi} = -K(1 + Hh)\phi \quad . \tag{3.17a,b}$$

Using the initial conditions (3.16c,d), the solution of (3.17) becomes

$$\frac{\theta - \theta_o}{V} = \exp(-Hh\tau) \quad , \quad \phi = 0 \quad , \tag{3.18a,b}$$

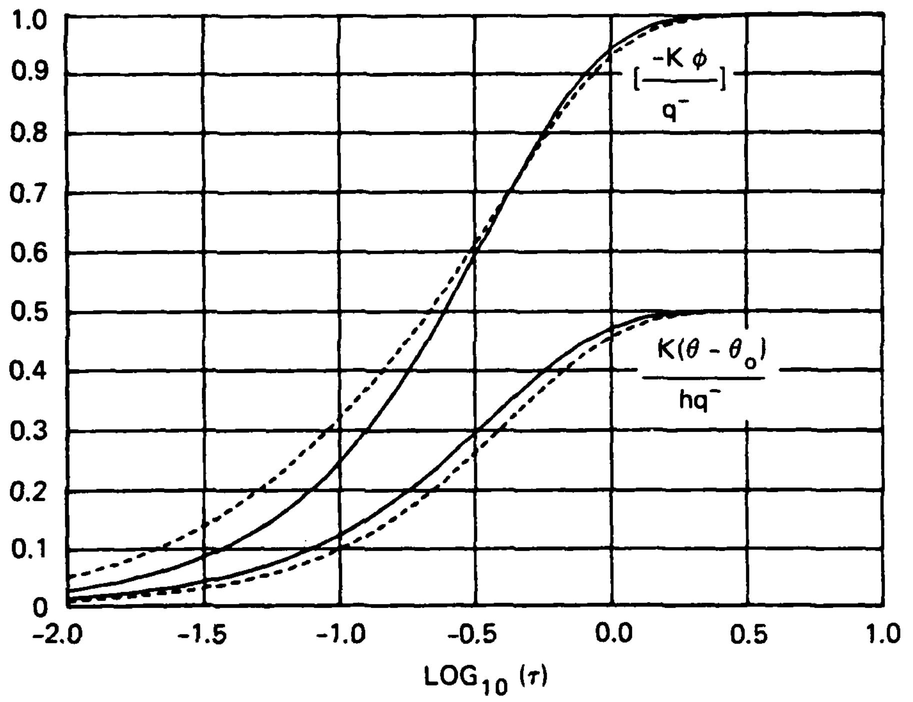

FIGURE D.3 VALUES OF THE NORMALIZED AVERAGE TEMPERATURE $[K(\theta - \theta_o)hq^-]$ AND AVERAGE TEMPERATURE GRADIENT $[-K\phi/q^-]$ FOR A PLATE OF THICKNESS h WITH HEAT FLUX q^- (ENTERING THE PLATE) AT THE BOTTOM SURFACE AND THE TEMPERATURE $\theta^+ = \theta_o$ SPECIFIED ON THE TOP SURFACE

Initially, the temperature $\theta^- = \theta_o$ at the bottom surface. The dashed lines are the exact solution and the solid lines are the Cosserat solution. $\tau = Kt/\rho_o^* ch^2$.

where τ is again defined by (3.6). To compare the Cosserat solution (3.18) with the exact solution recorded in [4, p. 122], we rewrite the exact solution in the form

$$\frac{\theta^* - \theta_o}{V} = \sum_{n=1}^{\infty} \frac{2\ Hh \cos\left(\frac{\alpha_n \theta^3}{h}\right) \sec \alpha_n}{\left[(Hh)^2 + Hh + \alpha_n^2\right]} \exp(-\alpha_n^2 \tau) \quad , \tag{3.19a}$$

$$\alpha_n \tan \alpha_n = Hh \quad , \tag{3.19b}$$

where α_n are the positive roots of equation (3.19b) and where $\theta^3 = 0$ locates the center of the plate, $\theta^3 = h$ locates the top surface, and $\theta^3 = -h$ locates the bottom surface. Replacing h by 2h in (3.14) and using (3.19), we deduce the expressions

$$\frac{\theta^*_{avg} - \theta_o}{V} = \sum_{n=1}^{\infty} \frac{2\ (Hh)^2}{\alpha_n^2 \left[(Hh)^2 + Hh + \alpha_n^2\right]} \exp(-\alpha_n^2 \tau) \quad , \tag{3.20a}$$

$$\phi^*_{avg} = 0 \quad . \tag{3.20b}$$

Comparing (3.18b) with (3.20b), we see that the Cosserat theory predicts the correct value for the average temperature gradient. To compare the prediction of the average temperature, we have plotted (3.18a) as the solid lines and (3.20a) as the dashed lines in Figure D.4, for three values of the normalized radiation coefficient Hh. From Figure D.4, we observe that for small values of Hh, the Cosserat theory predicts accurate results whereas for large values of Hh, its does not. This is because for small values of Hh, heat is radiated slowly away from the major surfaces of the plate, so that the temperature through the thickness of the plate is nearly uniform, as predicted by (3.18b). However, for large values of Hh, heat is radiated rapidly away from the plate and the through-the-thickness temperature gradient may be steep.

Mathematically, we may exploit the symmetry in the problem stated above and thus confine attention only to the upper half of the plate.

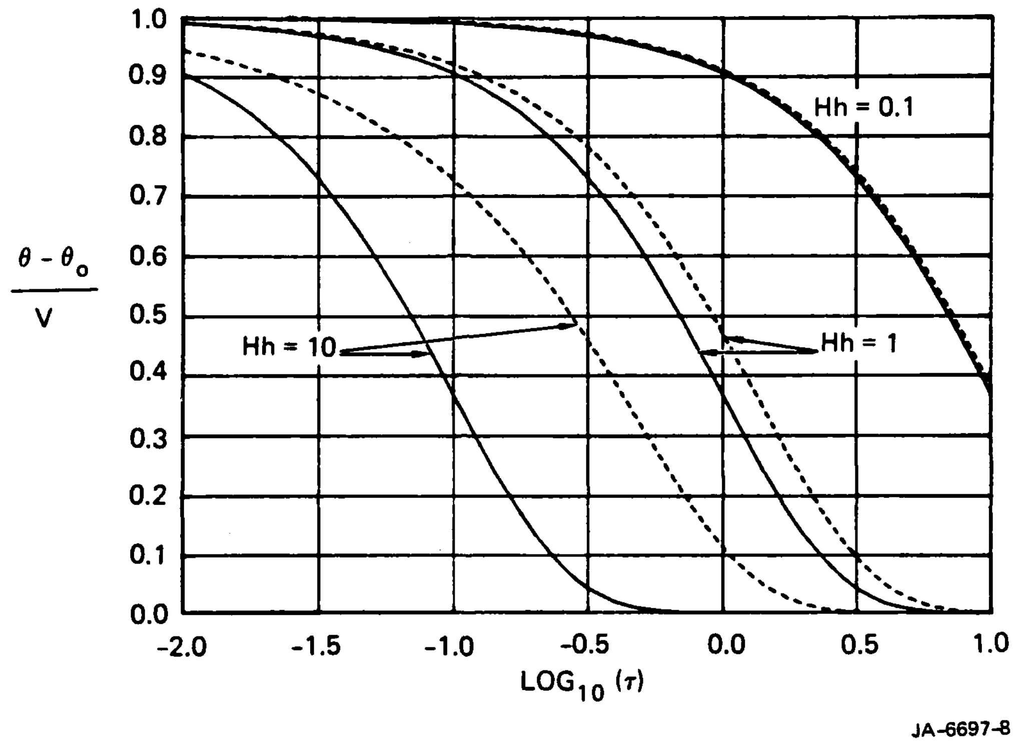

FIGURE D.4 VALUES OF THE NORMALIZED AVERAGE TEMPERATURE $(\theta - \theta_o)/V$ FOR A PLATE OF THICKNESS 2h WITH RADIATION FROM THE SURFACES [i.e. $q^+ = KH(\theta^+ - \theta_o)$] AND UNIFORM INITIAL TEMPERATURE $\theta = \theta_o + V$

The dashed lines are the exact solution and the solid lines are the Cosserat solution. $\tau = Kt/\rho_o^* ch^2$.

Therefore, for this second solution we consider a plate of thickness h. The heat flux is specified appropriately for radiation from the top surface, the bottom surface (which corresponds to the center surface of the plate of thickness 2h) is insulated, and the plate is initially at a uniform temperature $\theta_o + V$. These conditions are characterized by (3.16) with (3.16b) replaced by

$$q^{-} = 0 \quad . \tag{3.21}$$

At this point, it is important to note that although the exact solutions of the two problems considered here are identical, the Cosserat solution of the second problem will be more accurate than the Cosserat solution of the first problem. This is because the solution of the second problem admits a nonzero temperature gradient in the top half of the plate.

Substituting (2.16i), (3.16a), and (3.21) into equations (3.3), we obtain

$$\rho_o^* \, ch \, \dot{\theta} = - KH(\theta + \frac{h}{2} \phi - \theta_o) \quad , \tag{3.22a}$$

$$\frac{\rho_o^* \, ch^2}{\pi^2} \dot{\phi} = - \frac{1}{2} KH \left(\theta + \frac{h}{2} \phi - \theta_o\right) - K \phi \quad . \tag{3.22b}$$

In their present form these equations are coupled. However, by solving (3.22a) for ϕ and substituting the result into (3.22b), we may define

$$\frac{\theta - \theta_o}{V} = f(\tau) \quad , \tag{3.23}$$

and write

$$- \frac{h \phi}{V} = 2 f(\tau) + \frac{2}{Hh} \frac{df}{d\tau} \quad , \quad \frac{d^2 f}{d\tau^2} + B \frac{df}{d\tau} + C f = 0 \quad , \tag{3.24a,b}$$

where B and C are constants defined by

$$B = \pi^2 + Hh + \frac{\pi^2}{4} Hh \quad , \quad C = \pi^2 Hh \quad , \tag{3.25a,b}$$

and τ is defined by (3.6). Using (3.6), (3.23), and (3.24a), the initial conditions (3.16c,d) become

$$f = 1 \quad , \quad \frac{df}{d\tau} = - Hh \quad , \quad \text{at} \quad \tau = 0 \tag{3.26a,b}$$

Solving (3.24b) subject to the conditions (3.26), the Cosserat solution may be written in the form

$$\frac{\theta - \theta_o}{V} = A_1 \exp(-\sigma_1 \tau) + A_2 \exp(-\sigma_2 \tau) \quad , \tag{3.27a}$$

$$- \frac{h\,\phi}{V} = 2\,[A_1 \exp(-\sigma_1 \tau) + A_2 \exp(-\sigma_2 \tau)]$$

$$- \left(\frac{2}{Hh}\right)[A_1\,\sigma_1 \exp(-\sigma_1 \tau) + A_2\,\sigma_2 \exp(-\sigma_2 \tau)] \quad , \tag{3.27b}$$

where the constants A_1, A_2, σ_1, σ_2 are given by

$$A_1 = \frac{\sigma_2 - Hh}{\sigma_2 - \sigma_1} \quad , \quad A_2 = \frac{Hh - \sigma_1}{\sigma_2 - \sigma_1} \quad , \tag{3.28a,b}$$

$$\sigma_1 = \frac{1}{2}\,[B - (B^2 - 4\,C)^{1/2}] \quad , \quad \sigma_2 = \frac{1}{2}\,[B + (B^2 - 4\,C)^{1/2}] \quad . \tag{3.28,c,d}$$

Replacing θ^3 in (3.19a) by $h/2 + \theta^3$, we may write the exact solution for the top half of the plate as

$$\frac{\theta^* - \theta_o}{V} = \sum_{n=1}^{\infty} \frac{2\,Hh \cos\left\{\frac{\alpha_n (h + 2\,\theta^3)}{2\,h}\right\} \sec \alpha_n}{\left[(Hh)^2 + Hh + \alpha_n^2\right]} \exp(-\alpha_n^2\,\tau) \quad , \tag{3.29}$$

where α_n are the positive roots of (3.19b), and where $\theta^3 = h/2$ locates the top surface and $\theta^3 = - h/2$ locates the bottom surface (which corresponds to the center surface of the plate of thickness 2h). Substituting (3.29) into the definitions (3.14a,b), we obtain the result (3.20a) for the average temperature θ^*_{avg} and the result

$$- \frac{h \phi^*_{avg}}{V} = - 12 \sum_{n=1}^{\infty} \frac{Hh [\alpha_n \sin \alpha_n + 2 (\cos \alpha_n - 1)] \sec \alpha_n}{\alpha_n^2 [(Hh)^2 + Hh + \alpha_n^2]} \exp (- \alpha_n^2 \tau) \quad . \tag{3.30}$$

Figure D.5 compares values of the average temperature with (3.27a) plotted as the solid lines and (3.20a) plotted as the dashed lines. Similarly, Figure D.6 compares values of the average temperature gradient with (3.27b) plotted as the solid lines and (3.30) plotted as the dashed lines. From Figures D.4 and D.5, we observe that modeling only the upper half of the plate produces a significant improvement in the prediction of the average temperature for the higher values of Hh. Also, we observe from Figures D.5 and D.6 that for $Hh = 0.1$, the Cosserat and exact solutions are nearly identical and the average temperature gradient remains relatively small.

4. Circular Cylindrical Shells

In this section, we investigate the validity of the Cosserat theory in the thick-shell limit by considering heat conduction in a circular cylindrical shell and taking the limit of a solid cylinder. Specifically, consider a circular cylindrical shell of uniform thickness h and mean radius R. Let $\underset{\sim}{e}_i (i = 1,2,3)$ be a set of fixed Cartesian base vectors and let $\underset{\sim}{e}'_i$ be base vectors of a polar coordinate system with polar angle γ defined by*

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*Although this coordinate system is unconventional, it is chosen because it yields convenient relations between $\underset{\sim}{A}_i$ and $\underset{\sim}{e}'_i$.

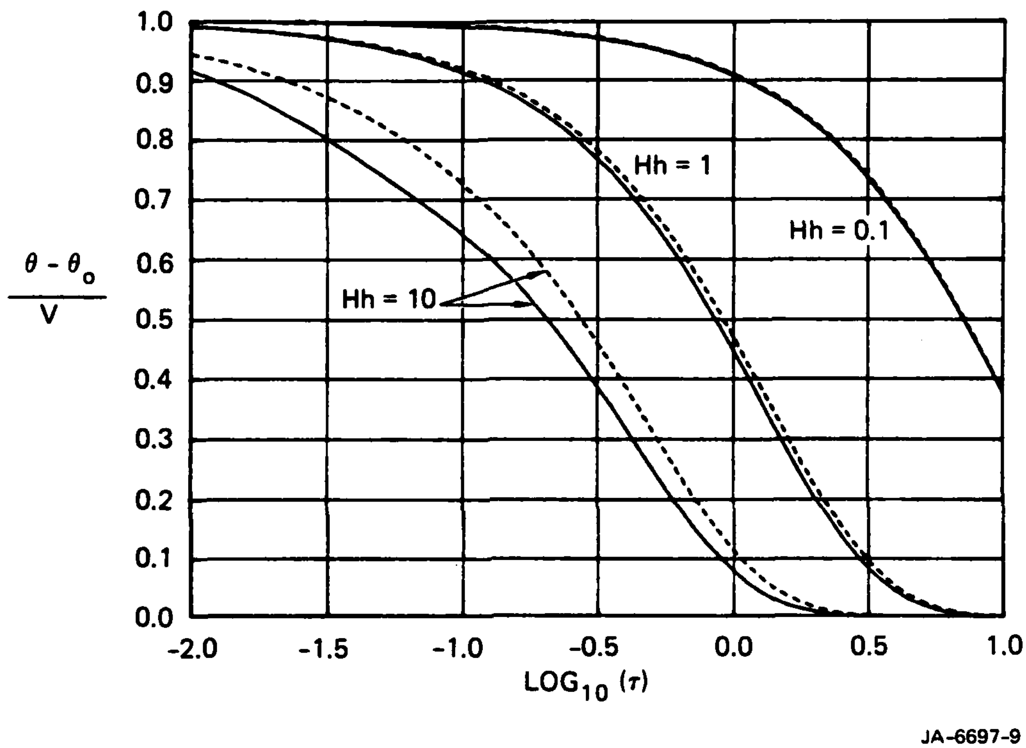

FIGURE D.5 VALUES OF THE NORMALIZED AVERAGE TEMPERATURE $(\theta - \theta_o)/V$ FOR A PLATE OF THICKNESS h WITH RADIATION FROM THE TOP SURFACE [i.e., $q^+ = KH(\theta^+ - \theta_o)$], ZERO HEAT FLUX ON THE BOTTOM SURFACE, AND UNIFORM INITIAL TEMPERATURE $\theta = \theta_o + V$

The dashed lines are the exact solution and the solid lines are the Cosserat solution. $\tau = Kt/\rho_o^* ch^2$.

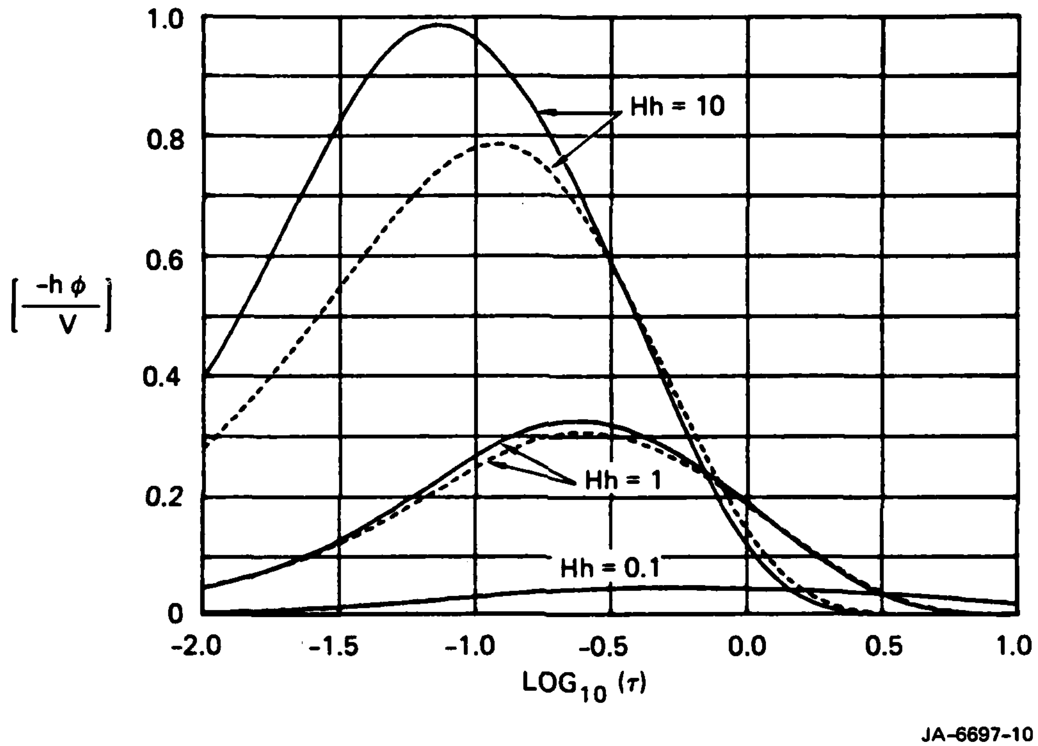

FIGURE D.6 VALUES OF THE NORMALIZED AVERAGE TEMPERATURE GRADIENT [$-h\,\phi/V$] FOR A PLATE OF THICKNESS h WITH RADIATION FROM THE TOP SURFACE [i.e., $q^+ = KH\,(\theta^+ - \theta_o)$], ZERO HEAT FLUX ON THE BOTTOM SURFACE, AND UNIFORM INITIAL TEMPERATURE $\theta = \theta_o + V$

The dashed lines are the exact solution and the solid lines are the Cosserat solution. $\tau = Kt/\rho_o^* \, ch^2$.

$$\underset{\sim}{e}'_1 = \underset{\sim}{e}_1 \quad , \quad \underset{\sim}{e}'_2 = \cos\gamma\, \underset{\sim}{e}_2 - \sin\gamma\, \underset{\sim}{e}_3 \quad , \quad \underset{\sim}{e}'_3 = \sin\gamma\, \underset{\sim}{e}_2 + \cos\gamma\, \underset{\sim}{e}_3 \quad , \tag{4.1a,b,c}$$

where $\underset{\sim}{e}'_1$ is parallel to the generator of the cylindrical geometry.

Now, points on the reference surface of the shell may be located by the position vector $\underset{\sim}{R}$ given by

$$\underset{\sim}{R} = x\, \underset{\sim}{e}'_1 + R\, \underset{\sim}{e}'_3 \quad , \quad \theta^1 = x \quad , \quad \theta^2 = \gamma \quad , \tag{4.2a,b,c}$$

where we have identified the coordinates θ^1, θ^2 with x and γ, respectively. Using the definitions in [2] and in Section 2, the relevant geometrical properties of the cylindrical surface may be recorded as

$$A^{1/2} = R \quad , \quad A^{11} = 1 \quad , \quad A^{12} = 0 \quad , \quad A^{22} = \frac{1}{R^2} \quad , \tag{4.3a,b,c,d}$$

$$B^2_2 = -\frac{1}{R} \quad , \quad \text{all other} \quad B^\alpha_\beta = 0 \quad , \quad \Gamma^\sigma_{\alpha\beta} = 0 \quad . \tag{4.3e,f,g}$$

Substituting (4.3) into (2.16), we have

$$\rho_o = \rho^*_o h \quad , \quad B^+ = A^{1/2}\left(1 + \frac{h}{2R}\right) \quad , \quad B^- = A^{1/2}\left(1 - \frac{h}{2R}\right) \quad . \tag{4.4a,b,c}$$

It follows that the thermal equations (2.19) become

$$\rho^*_o ch\, \dot{\theta} = \rho^*_o h\, \theta_o\, \hat{s} - \left(1 + \frac{h}{2R}\right) q^+ + \left(1 - \frac{h}{2R}\right) q^- + Kh\, \nabla^2 \theta \quad , \tag{4.5a}$$

$$\frac{\rho^*_o ch^2}{\pi^2} \dot{\phi} = \rho^*_o\, \theta_o\, \hat{s}_1 - \frac{1}{2}\left(1 + \frac{h}{2R}\right) q^+ - \frac{1}{2}\left(1 - \frac{h}{2R}\right) q^-$$

$$- K\phi + \frac{Kh^2}{12} \nabla^2 \phi \quad , \tag{4.5b}$$

where the Laplacian operator $\nabla^2 \theta$ for the cylindrical geometry is given by

$$\nabla^2 \theta = \frac{\partial^2 \theta}{\partial x^2} + \frac{1}{R^2} \frac{\partial^2 \theta}{\partial \gamma^2} \quad . \tag{4.6}$$

Here, we consider the problem for which the heat flux on the outer surface is constant, the inner surface is insulated, external entropy supply is neglected, and the shell is initially at uniform temperature θ_o. Hence, the conditions (3.1) and (3.4) hold and equations (4.5) reduce to

$$\rho_o^* \, ch \, \dot{\theta} = - \left(1 + \frac{h}{2R}\right) q^+ \quad , \tag{4.7a}$$

$$\rho_o^* \, ch \, \dot{\phi} = - K \, \phi - \frac{1}{2} \left(1 + \frac{h}{2R}\right) q^+ \quad . \tag{4.7b}$$

Integrating (4.7) subject to the initial conditions (3.4c,d), we obtain

$$- \frac{K(\theta - \theta_o)}{h \, q^+} = \left(1 + \frac{h}{2R}\right) \tau \quad , \tag{4.8a}$$

$$- \frac{K \, \phi}{q^+} = \frac{1}{2} \left(1 + \frac{h}{2R}\right)[1 - \exp\,(- \pi^2 \, \tau)] \quad , \tag{4.8b}$$

where τ is defined by (3.6). Notice that in the thin-shell limit ($R/h \to \infty$), the solution (4.8) approaches the plate solution (3.5). In the thick-shell limit of a solid cylinder for which* $R = h/2$, the right-hand side of (4.8a) becomes 2τ, which is consistent with the exact solution [4, p. 203]. Using (2.15b), the Cosserat solution (4.8) with $R = h/2$, is plotted in Figure D.7 together with the exact solution for various values of the time parameter τ. The dashed lines in Figure D.7 have been taken directly from [4, Figure 25, p. 203] and r is the radial

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*From [2], we recall that generally $G^{1/2}$ is required to be positive. Although the quantity $G^{1/2}$ vanishes when $\theta^3 = - h/2$ and $R = h/2$, this poses no particular difficulty in the problems considered here.

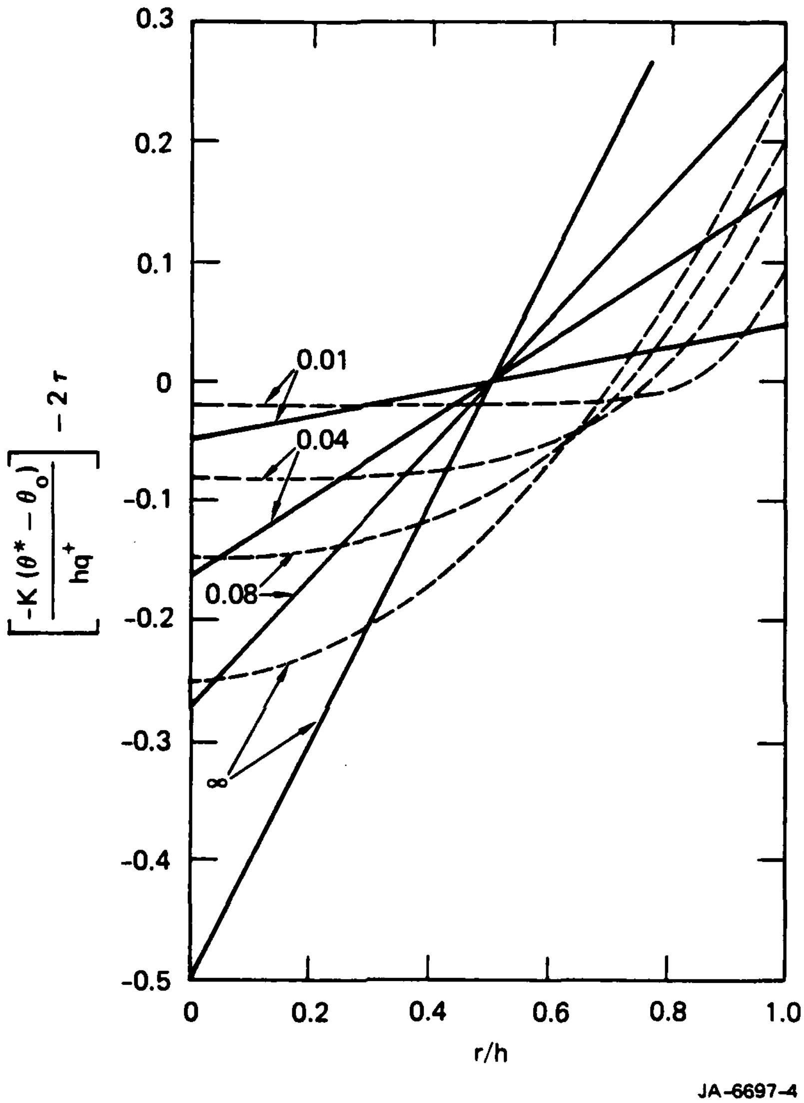

FIGURE D.7 NORMALIZED TEMPERATURE IN A SOLID CIRCULAR CYLINDER OF RADIUS h, WITH CONSTANT HEAT FLUX q^+ (OUT OF THE CYLINDER) AT THE SURFACE, AND UNIFORM INITIAL TEMPERATURE $\theta = \theta_o$

The numbers on the curved lines are values of $\tau = Kt/\rho_o^* ch^2$. The dashed lines are the exact solution and the solid lines are the Cosserat solution.

coordinate, with $r = 0$ locating the center of the cylinder and $r = h$ locating the outer surface.

In view of the form of solution (4.8), it is obvious that the long-time temperature is dominated by the term (4.8a). The fact that the coefficient of τ in (4.8a) yields the correct result even in the thick-shell limit suggests that the Cosserat theory retains the important geometrical features of the shell. In this regard, it is worth mentioning that the result (4.8a) could be obtained using an engineering approach in which the temperature in the shell is assumed to be uniform and the energy entering the outer surface is equated with the increase in internal energy. It is also worth mentioning that the solution of the more accurate equation (15) of [1] yields a long-time solution of the form

$$-\frac{K(\theta^* - \theta_o)}{h\, q^+} \rightarrow \left[\frac{\sinh\left(\frac{h}{2R}\right) + \cosh\left(\frac{h}{2R}\right)}{\frac{2R}{h}\sinh\left(\frac{h}{2R}\right)}\right]\tau \quad . \tag{4.9}$$

In the thin-shell limit (4.9) yields the correct result, but in the thick-shell limit it yields the result (2.313 τ), which is incorrect. Thus, even though equation (15) of [1] is more complicated than equations (4.8), it does not necessarily produce a better result.

To further examine the validity of the constitutive equations (2.14b,c) and the specifications (2.18a,c), we consider a simple problem for which the Laplacian operators in (4.5) do not vanish. Specifically, consider the steady-state problem of uniform heat conduction in the constant $\mathbf{e}_3$ direction for which the three-dimensional solution is given by

$$\mathbf{q}^* = Q\,\mathbf{e}_3 \quad , \quad \theta^* = \left(\theta_o - \frac{QR}{K}\cos\gamma\right) - \theta^3\left(\frac{Q}{K}\cos\gamma\right) \tag{4.10a,b}$$

where $\mathbf{q}^*$ is the three-dimensional heat conduction vector and Q is a constant. Using (4.1) and (4.10), we realize that

$$q^+ = q^- = \underset{\sim}{q}^* \cdot \underset{\sim}{e}_3' = Q \cos\gamma \quad . \tag{4.11}$$

Consequently, in the absence of external entropy supply, the steady-state solution of (4.5) becomes

$$\theta = \theta_o - \frac{QR}{K}\cos\gamma \quad , \quad \phi = -\left(1 + \frac{h^2}{12R^2}\right)^{-1} \frac{Q}{K}\cos\gamma \quad . \tag{4.12a,b}$$

Now, with the help of (2.15b) we may compare the exact result (4.10b) with the Cosserat result (4.12) to conclude that the average temperature is predicted exactly. Further, the prediction of the average temperature gradient ϕ is very accurate in the thin-shell limit ($R/h \to \infty$) and is only 25% low in the thick-shell limit ($R/h \to 1/2$).

5. Spherical Shells

The spherical shell geometry is considered here mainly because it is one of the simplest geometries in which it is possible to investigate the differences between the constitutive assumptions (2.12) and (2.14). Three problems of a spherical shell of constant thickness h and radius R are considered. For the first two problems, we consider the thick-shell limit of a solid sphere and discuss the differences between assumptions (2.12) and (2.14). For the third problem, we consider the transition from a thick shell to a thin shell.

With reference to the Cartesian base vectors $\underset{\sim}{e}_i$ introduced in Section 4, we let $\underset{\sim}{e}_i''$ be base vectors of a spherical coordinate system with polar angle γ ($0 \le \gamma \le 2\pi$) measured from the $\underset{\sim}{e}_1 - \underset{\sim}{e}_3$ plane and polar angle σ $\left(-\frac{\pi}{2} \le \sigma \le \frac{\pi}{2}\right)$ measured from the $\underset{\sim}{e}_1 - \underset{\sim}{e}_2$ plane such that*

$$\underset{\sim}{e}_1'' = -\sin\gamma\, \underset{\sim}{e}_1 + \cos\gamma\, \underset{\sim}{e}_2 \quad , \tag{5.1a}$$

---

*Although this coordinate system is unconventional, it is chosen because it yields convenient relations between $\underset{\sim}{A}_i$ and $\underset{\sim}{e}_i''$.

$$\underset{\sim}{e}_2'' = -\sin\sigma\,(\cos\gamma\,\underset{\sim}{e}_1 + \sin\gamma\,\underset{\sim}{e}_2) + \cos\sigma\,\underset{\sim}{e}_3 \quad , \tag{5.1b}$$

$$\underset{\sim}{e}_3'' = \cos\sigma\,(\cos\gamma\,\underset{\sim}{e}_1 + \sin\gamma\,\underset{\sim}{e}_2) + \sin\sigma\,\underset{\sim}{e}_3 \quad . \tag{5.1c}$$

Now, points on the reference surface of the shell may be located by the position vector $\underset{\sim}{R}$ given by

$$\underset{\sim}{R} = R\,\underset{\sim}{e}_3'' \quad , \quad \theta^1 = \gamma \quad , \quad \theta^2 = \sigma \quad , \tag{5.2a,b,c}$$

where we have identified the coordinates θ^1, θ^2 with γ and σ, respectively. Using the definitions in [2] and in Section 2, the relevant geometrical properties of the spherical surface may be recorded as

$$A^{1/2} = R^2 \cos\sigma \quad , \quad A^{11} = \frac{1}{R^2 \cos^2\sigma} \quad , \quad A^{12} = 0 \quad , \quad A^{22} = \frac{1}{R^2} \quad , \tag{5.3a,b,c,d}$$

$$B^1_1 = B^2_2 = -\frac{1}{R} \quad , \quad \text{all other} \quad B^\alpha_\beta = 0 \quad , \tag{5.3e,f}$$

$$\Gamma^1_{12} = \Gamma^1_{21} = -\tan\sigma \quad , \quad \Gamma^2_{11} = \sin\sigma\cos\sigma \quad , \quad \text{all other} \quad \Gamma^\sigma_{\alpha\beta} = 0 \quad . \tag{5.3g,h,i}$$

Substituting (5.3) into (2.16), we have

$$\rho_o = \rho_o^* h \left(1 + \frac{h^2}{12\,R^2}\right) \quad , \quad B^+ = A^{1/2}\left(1 + \frac{h}{2R}\right)^2 \quad , \quad B^- = A^{1/2}\left(1 - \frac{h}{2R}\right)^2 \quad . \tag{5.4a,b,c}$$

It follows that the thermal equations (2.19) become

$$\rho_o^*\, ch \left(1 + \frac{h^2}{12\,R^2}\right) \dot{\theta} = \rho_o\,\theta_o\,\hat{s} - \left(1 + \frac{h}{2R}\right)^2 q^+ + \left(1 - \frac{h}{2R}\right)^2 q^- + Kh\,\nabla^2\theta \tag{5.5a}$$

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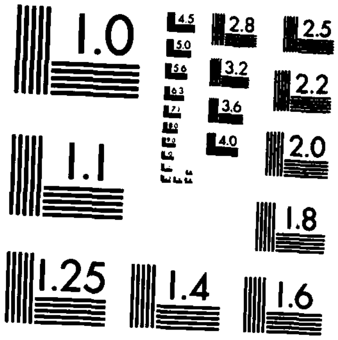

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$$\frac{\rho_o^* ch^3}{\pi^2}\left(1 + \frac{h^2}{12\ R^2}\right)\dot{\phi} = \rho_o\ \theta_o\ \hat{s}_1 - \left(\frac{h}{2}\right)\left(1 + \frac{h}{2R}\right)^2 q^+ - \left(\frac{h}{2}\right)\left(1 - \frac{h}{2R}\right)^2 q^-$$

$$- Kh\left(1 + \frac{h^2}{12\ R^2}\right)\phi + \frac{Kh^3}{12}\nabla^2\phi \quad , \tag{5.5b}$$

where the Laplacian operator $\nabla^2\ \theta$ for the spherical geometry is given by

$$\nabla^2\ \theta = \frac{1}{R^2\ \cos^2\sigma}\frac{\partial^2\theta}{\partial\gamma^2} + \frac{1}{R^2}\frac{\partial^2\theta}{\partial\sigma^2} - \frac{\tan\sigma}{R^2}\frac{\partial\theta}{\partial\sigma}\ . \tag{5.6}$$

For the first problem, the heat flux on the outer surface is constant, the inner surface is insulated, external entropy supply is neglected, and the shell is initially at uniform temperature θ_o. Hence, the conditions (3.1) and (3.4) hold and equations (5.5) reduce to

$$\rho_o^*\ ch\ \ 1 + \frac{h^2}{12\ R^2}\ \ \dot{\theta} = -\left(1 + \frac{h}{2R}\right)^2 q^+ \quad , \tag{5.7a}$$

$$\frac{\rho_o^* ch^2}{\pi^2}\left(1 + \frac{h^2}{12\ R^2}\right)\dot{\phi} = -\frac{1}{2}\left(1 + \frac{h}{2R}\right)^2 q^+ - K\left(1 + \frac{h^2}{12\ R^2}\right)\phi\ . \tag{5.7b}$$

Integrating (5.7) subject to the initial conditions (3.4c,d), we obtain

$$-\frac{K(\theta - \theta_o)}{h\ q^+} = \frac{\left(1 + \frac{h}{2R}\right)^2}{\left(1 + \frac{h^2}{12\ R^2}\right)}\tau \quad , \tag{5.8a}$$

$$-\frac{K\ \phi}{q^+} = \frac{\left(1 + \frac{h}{2R}\right)^2}{2\left(1 + \frac{h^2}{12\ R^2}\right)}\left[1 - \exp\left(-\pi^2\ \tau\right)\right] \quad , \tag{5.8b}$$

where τ is defined by (3.6). Notice that in the thin-shell limit ($R/h \to \infty$) the solution (5.8) approaches the plate solution (3.5). In the thick-shell limit of a solid sphere for which $R = h/2$, the right-

hand side of (5.8a) becomes 3τ, which is consistent with the exact solution [4, p. 242]. Using (2.15b), the Cosserat solution (5.8) with $R = h/2$ is plotted in Figure D.8 together with the exact solution for various values of the time parameter τ. The dashed lines in Figure D.8 have been taken directly from [4, Figure 31, p. 242] and r is the radial coordinate with $r = 0$ locating the center of the sphere and $r = h$ locating the outer surface.

From Figure D.8, we observe that for long time periods the value of the average temperature gradient predicted by the Cosserat theory is substantially larger than the exact value. However, this is not particularly important because for long times the temperature is dominated by the term (5.8a). To exhibit this, we have used (2.16i,j) together with (5.8) to plot in Figure D.9 the temperature on the outer surface and at the center of the solid sphere. The dashed lines in Figure D.9 represent the exact solution [4, p. 242]. For short times, the Cosserat theory predicts the incorrect result that the center temperature of the sphere drops. This is a consequence of the over-prediction of the average temperature gradient. For long times, the lines in Figure D.9 are parallel and the relative error diminishes to zero. This is because the prediction (5.8a) is exact in the thick-shell limit. In this regard, it is worth mentioning that the result (5.8a) could be obtained using the engineering approach described in Section 4. It is also worth mentioning that the more accurate equation (15) of [1] yields a long-time solution of the form*

$$-\frac{K(\theta^* - \theta_o)}{h\,q^+} \to \left[\frac{\sinh\left(\frac{h}{R}\right) + \cosh\left(\frac{h}{R}\right)}{\left(\frac{h}{R}\right)\sinh\left(\frac{h}{R}\right)}\right]\tau \quad , \tag{5.9}$$

*The solution in Appendix A of [1] should be written in a form that has a linear term in time.

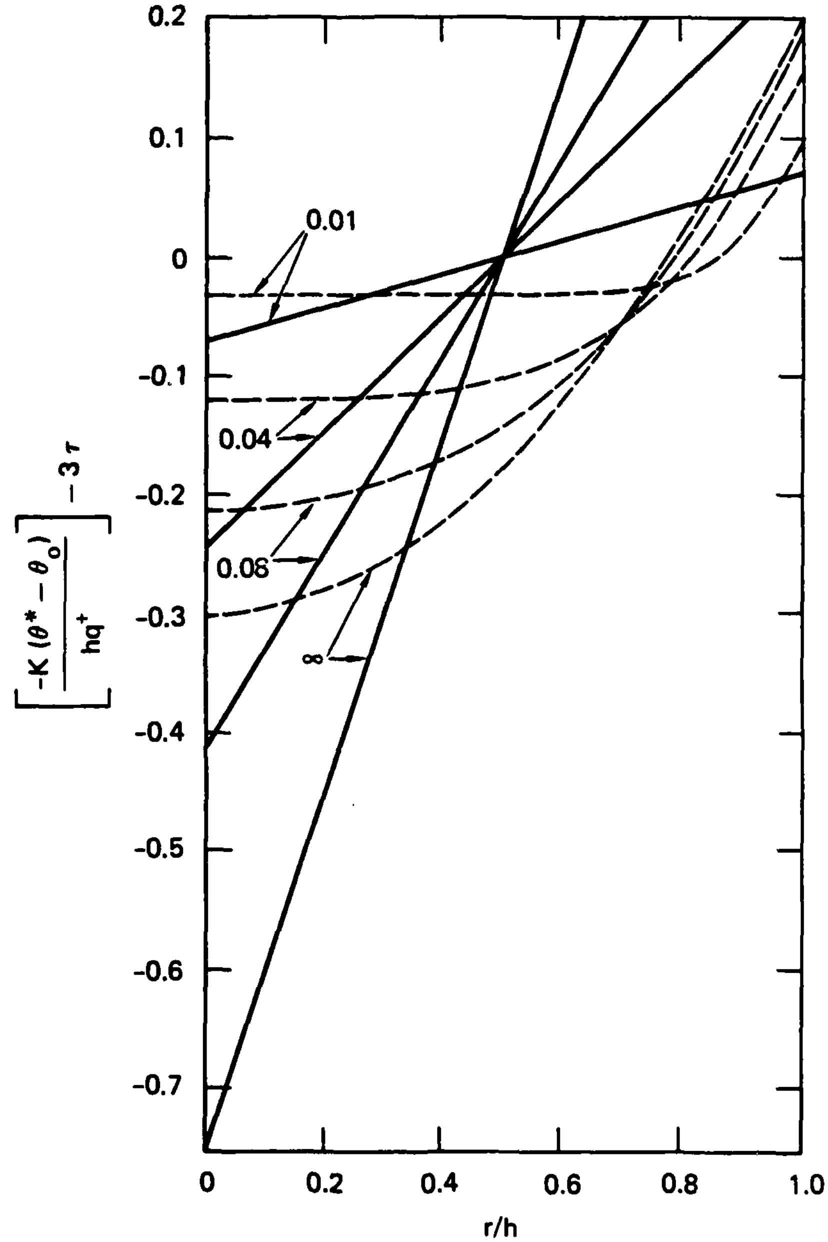

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FIGURE D.8 NORMALIZED TEMPERATURE IN A SOLID SPHERE OF RADIUS h, WITH CONSTANT HEAT FLUX q^+ (OUT OF THE SPHERE) AT THE SURFACE, AND UNIFORM INITIAL TEMPERATURE $\theta = \theta_o$

The numbers on the curves are values of $\tau = Kt/\rho_o^* ch^2$. The dashed lines are the exact solution and the solid lines are the Cosserat solution.

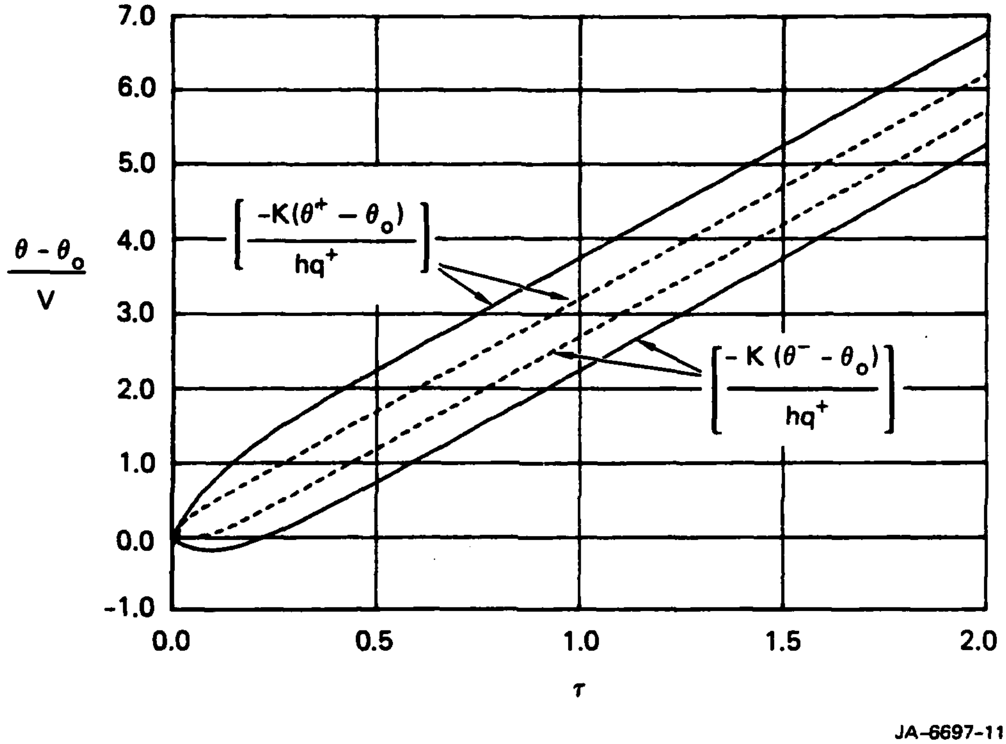

FIGURE D.9 NORMALIZED TEMPERATURE $[-K(\theta^+ - \theta_o)/hq^+]$ AT THE SURFACE AND $[-K(\theta^- - \theta_o)/hq^+]$ AT THE CENTER OF A SOLID SPHERE OF RADIUS h, WITH CONSTANT HEAT FLUX q^+ (OUT OF THE SPHERE) AT THE SURFACE, AND UNIFORM INITIAL TEMPERATURE $\theta = \theta_o$

The dashed lines are the exact solution and the solid lines are the Cosserat solution. $\tau = Kt/\rho_o^* ch^2$.

In the thin-shell limit (5.9) yields the correct result, but in the thick-shell limit it yields the result (4.075 τ), which is incorrect.

We are now in a position to comment on the differences between the constitutive equations (2.12) and (2.14). If (2.12a) were used instead of (2.14a), then the average temperature would be given by

$$-\frac{K(\theta - \theta_o)}{h\, q^+} = \left(1 + \frac{h}{2R}\right)^2 \tau \tag{5.10}$$

instead of (5.8a). This would yield the incorrect result 4τ in the thick-shell limit. Similarly, if (2.12d,e) were used instead of (2.14d,e), then the long time value of ϕ would be

$$-\frac{K\,\phi}{q^+} = \frac{1}{2}\left(1 + \frac{h}{2R}\right)^2 \tag{5.11}$$

which produces a larger error than that associated with (5.8b) in the thick-shell limit.

To further examine the validity of constitutive equations (2.14b,c) and the specifications (2.18a,c), we consider the steady-state problem of uniform heat conduction in the constant $\underset{\sim}{e}_1$ direction for which the three-dimensional solution is given by

$$\underset{\sim}{q}^* = Q\,\underset{\sim}{e}_1 \quad , \quad \theta^* = \left(\theta_o - \frac{QR}{K}\cos\gamma\cos\sigma\right) - \theta^3\left(\frac{Q}{K}\cos\gamma\cos\sigma\right) \quad , \tag{5.12a,b}$$

where $\underset{\sim}{q}^*$ is the three-dimensional heat conduction vector and Q is a constant. Using (5.1) and (5.12), we realize that

$$q^+ = q^- = \underset{\sim}{q}^* \cdot \underset{\sim}{e}_3'' = Q\cos\gamma\cos\sigma \quad . \tag{5.13}$$

Consequently, in the absence of external entropy supply the steady-state solution of (5.5) becomes

$$\theta = \theta_o - \frac{QR}{K}\cos\gamma\cos\sigma \quad , \quad \phi = -\frac{Q}{K}\cos\gamma\cos\sigma \quad , \qquad (5.14a,b)$$

which is an exact result valid for both the thin- and thick-shell limits.

Finally, to examine the transition from a thin-shell to a thick-shell, we consider the steady-state problem where the temperature θ^+ on the outer surface is specified to be the constant value θ_o and the heat flux q^- on the inner surface is constant. Thus, using (2.16i) we require

$$\theta + \frac{h}{2}\phi = \theta_o \quad , \quad q^- = \text{constant} \quad . \qquad (5.15a,b)$$

In the absence of external entropy supply, the steady-state solution of (5.5) becomes

$$\frac{K(\theta - \theta_o)}{h\, q^-} = \frac{(1 - \frac{h}{2R})^2}{2(1 + \frac{h^2}{12\, R^2})} \quad , \quad \frac{K\,\phi}{q^-} = -\frac{(1 - \frac{h}{2R})^2}{(1 + \frac{h^2}{12\, R^2})} \quad , \qquad (5.16a,b)$$

$$\frac{q^+}{q^-} = \frac{(1 - \frac{h}{2R})^2}{(1 + \frac{h}{2R})^2} \quad . \qquad (5.16c)$$

It can be shown that the exact solution [4, p. 247] may be written in the form

$$\frac{K(\theta^* - \theta_o)}{h\, q^-} = \frac{(1 - \frac{h}{2R})^2(1 - \frac{2\,\theta^3}{h})}{2(1 + \frac{h}{2R})(1 + \frac{\theta^3}{R})} \quad , \qquad (5.17)$$

and that (5.16c) is an exact result. Now to compare the predictions (5.16) with the exact solution (5.17), we have used (2.15b) to plot (5.16) as the solid lines in Figure D.10 and have used (5.17) to plot the dashed lines in Figure D.10 for three values of R/h. The results in Figure D.10 show again that the Cosserat predictions are good even for a fairly thick shell (R/h = 1).

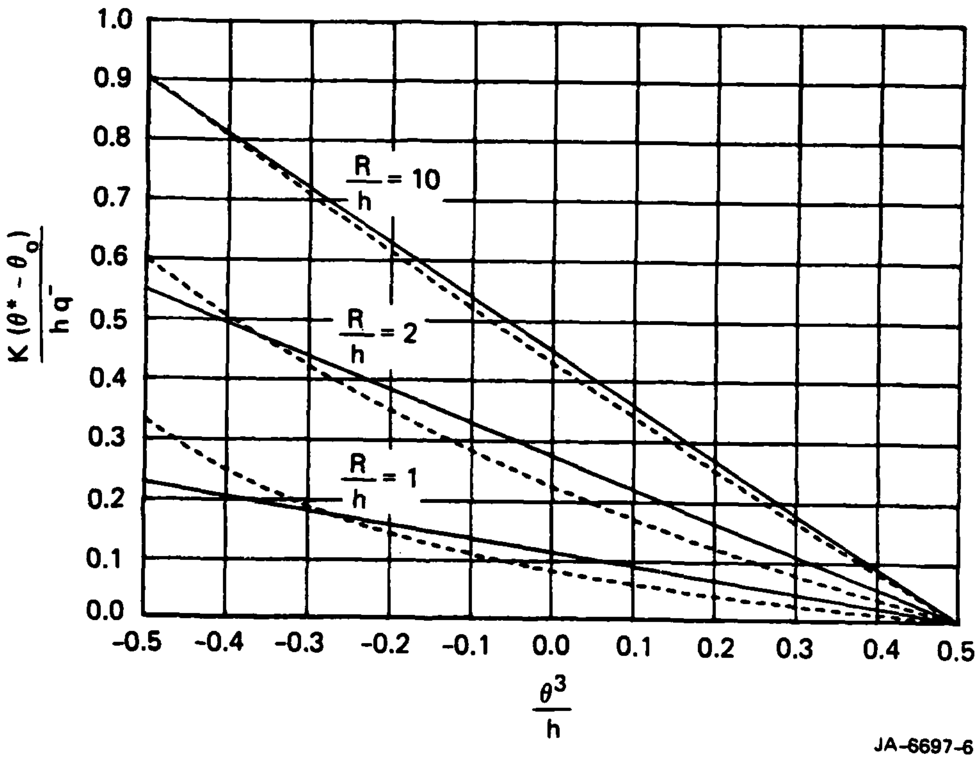

FIGURE D.10 THE NORMALIZED STEADY STATE TEMPERATURE IN A SPHERICAL SHELL WITH CONSTANT THICKNESS h AND MEAN RADIUS R

The heat flux q^- (entering the shell) is specified on the inner surface and the temperature $\theta^+ = \theta_o$ is specified on the outer surface. The dashed lines are the exact solution and the solid lines are the Cosserat solution.

6. Conical Shell

In the previous sections, we have solved a number of problems for plates, circular cylindrical shells, and spherical shells to develop confidence that the Cosserat theory can predict relatively accurate results for both the thin-shell limit (which models the base of a conical shell) and the thick-shell limit (which models the tip of a conical shell). Here, we confine attention to a conical shell with constant thickness h and locate points on the conical surface by

$$\underset{\sim}{R} = \beta R \underset{\sim}{e}'_1 + R \underset{\sim}{e}'_3 \quad , \quad \theta^1 = R \quad , \quad \theta^2 = \gamma \quad , \tag{6.1a,b,c}$$

where R is the radial coordinate, γ is the polar angle, $\underset{\sim}{e}'_i$ are defined by (4.1), β is a constant related to the cone angle (see Figure D.1), and we have identified the coordinates θ^1 and θ^2 with R and γ, respectively. Using the definitions in [2] and in Section 2, the relevant geometrical properties of the conical surface may be recorded as

$$A^{1/2} = R(1+\beta^2)^{1/2} \quad , \quad A^{11} = \frac{1}{(1+\beta^2)} \quad , \quad A^{12} = 0 \quad , \quad A^{22} = \frac{1}{R^2} \quad , \tag{6.2a,b,c,d}$$

$$B^2_2 = -\frac{\beta}{R(1+\beta^2)^{1/2}} \quad , \quad \text{all other} \quad B^\alpha_\beta = 0 \quad , \tag{6.2e,f}$$

$$\Gamma^1_{22} = -\frac{R}{(1+\beta^2)} \quad , \quad \Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{R} \quad , \quad \text{all other} \quad \Gamma^\sigma_{\alpha\beta} = 0 \quad . \tag{6.2g,h}$$

Substituting (6.2) into (2.16), we have

$$\rho_o = \rho^*_o h \quad , \quad B^+ = A^{1/2}\left[1 + \frac{h\,\beta}{2R(1+\beta^2)^{1/2}}\right] \quad , \quad B^- = A^{1/2}\left[1 - \frac{h\,\beta}{2R(1+\beta^2)^{1/2}}\right] \quad . \tag{6.3a,b,c}$$

It follows that the thermal equations (2.19) become

$$\rho_o^* ch \dot{\theta} = \rho_o^* h \theta_o \hat{s} - [1 + \frac{h \beta}{2R(1 + \beta^2)^{1/2}}] q^+ + [1 - \frac{h \beta}{2R(1 + \beta^2)^{1/2}}] q^-$$

$$+ K h \nabla^2 \theta \quad , \tag{6.4a}$$

$$\frac{\rho_o^* ch^2}{\pi^2} \dot{\phi} = \rho_o^* \theta_o \hat{s}_1 - \frac{1}{2} [1 + \frac{h \beta}{2R(1 + \beta^2)^{1/2}}] q^+ - \frac{1}{2} [1 - \frac{h \beta}{2R(1 + \beta^2)^{1/2}}] q^-$$

$$- K \phi + \frac{Kh^2}{12} \nabla^2 \phi \quad , \tag{6.4b}$$

where the Laplacian operator $\nabla^2 \theta$ for the conical geometry is given by

$$\nabla^2 \theta = \frac{1}{(1 + \beta^2)} [\frac{1}{R} \frac{\partial}{\partial R} (R \frac{\partial \theta}{\partial R}) + \frac{(1 + \beta^2)}{R^2} \frac{\partial^2 \theta}{\partial \gamma^2}] \quad . \tag{6.5}$$

Here, we consider the problem for which the heat flux on the outer surface is constant, all other surfaces are insulated, external entropy supply is neglected, and the shell is initially at uniform temperature θ_o. Hence, the conditions (3.4) hold in addition to the conditions

$$\theta = \theta(R,t) \quad , \quad \phi = \phi(R,t) \quad , \tag{6.6a,b}$$

$$\frac{\partial \theta}{\partial R} = 0 \quad , \quad \frac{\partial \phi}{\partial R} = 0 \quad \text{at} \quad R = R_1, R_2 \tag{6.6c,d}$$

where R_1 and R_2 are the tip radius and base radius of the shell, respectively (see Figure D.1). Under these conditions, equations (6.4) reduce to

$$\rho_o^* ch \dot{\theta} = - [1 + \frac{h \beta}{2R(1 + \beta^2)^{1/2}}] q^+ + \frac{Kh}{R(1 + \beta^2)} \frac{\partial}{\partial R} (R \frac{\partial \theta}{\partial R}) \quad , \tag{6.7a}$$

$$\frac{\rho_o^* ch^2}{\pi^2}\dot{\phi} = -\frac{1}{2}\left[1 + \frac{h\,\beta}{2R(1+\beta^2)^{1/2}}\right] q^+ - K\,\phi + \frac{Kh^2}{12R(1+\beta^2)}\frac{\partial}{\partial R}\left(R\,\frac{\partial\phi}{\partial R}\right) . \tag{6.7b}$$

To analyze these equations, it is convenient to introduce the nondimensional parameters

$$z = \frac{R}{h} \quad , \quad \tau = \frac{Kt}{\rho_o^*\, ch^2} \quad , \tag{6.8a,b}$$

$$\bar{\theta} = \bar{\theta}(z,\tau) = -\frac{K(\theta - \theta_o)}{h\, q^+} \quad , \quad \bar{\phi} = \bar{\phi}(z,\tau) = -\frac{K\phi}{q^+} \quad , \tag{6.8c,d}$$

and rewrite them in the form

$$\frac{\partial\bar{\theta}}{\partial\tau} - \frac{1}{(1+\beta^2)}\frac{1}{z}\frac{\partial}{\partial z}\left(z\,\frac{\partial\bar{\theta}}{\partial z}\right) = \left[1 + \frac{\beta}{2z(1+\beta^2)^{1/2}}\right] \quad , \tag{6.9a}$$

$$\frac{\partial\bar{\phi}}{\partial\tau} + \pi^2\,\bar{\phi} - \frac{\pi^2}{12(1+\beta^2)}\frac{1}{z}\frac{\partial}{\partial z}\left(z\,\frac{\partial\bar{\phi}}{\partial z}\right) = \frac{\pi^2}{2}\left[1 + \frac{\beta}{2z(1+\beta^2)^{1/2}}\right] \quad . \tag{6.9b}$$

Similarly, the initial conditions (3.4c,d) and boundary conditions (6.6c,d) become

$$\bar{\theta} = 0 \quad , \quad \bar{\phi} = 0 \quad \text{at} \quad \tau = 0 \tag{6.10a,b}$$

$$\frac{\partial\bar{\theta}}{\partial z} = 0 \quad , \quad \frac{\partial\bar{\phi}}{\partial z} = 0 \quad \text{at} \quad z = z_1, z_2 \quad , \tag{6.10c,d}$$

where z_1 and z_2 are the values of z when R equals R_1 and R_2, respectively. At this point, it is of interest to note that in the limit of large $\beta(\beta \to \infty)$, equations (6.9) reduce to a nondimensional form of (4.7) for a circular cylindrical shell, and in the limit of small $\beta(\beta \to 0)$, equations (6.9) characterize a circular plate.

Using standard techniques, the solution of (6.9) may be written in the form

$$\bar{\theta} = A_o(\tau) + \sum_{m=1}^{\infty} A_m(\tau)\, f_m(z) \quad , \tag{6.11a}$$

$$\bar{\phi} = B_o(\tau) + \sum_{m=1}^{\infty} B_m(\tau)\, f_m(z) \quad , \tag{6.11b}$$

where $f_m(z)$ are eigenfunctions characterized by

$$\frac{1}{z}\frac{d}{dz}\left(z \frac{df_m}{dz}\right) = -\alpha_m^2 f_m \qquad \text{(no sum on m)} \tag{6.12a}$$

$$\frac{df_m}{dz} = 0 \qquad \text{at} \quad z = z_1, z_2 \quad , \tag{6.12b}$$

and where α_n^2 are the nonzero eigenvalues. Since equation (6.12a) can easily be recognized as Bessel's equation of order zero, the solution, subject to the boundary conditions (6.12b), is well characterized and may be written in the form

$$f_m(z) = J_o(\alpha_m z) - \frac{J_1(\alpha_m z_1)}{Y_1(\alpha_m z_1)} Y_o(\alpha_m z) \tag{6.13}$$

where J_n and Y_n are Bessel functions of the first and second kind, respectively, of order n and where α_m are the positive roots of the characteristic equation

$$J_1(\alpha_m z_1)Y_1(\alpha_m z_2) - Y_1(\alpha_m z_1)J_1(\alpha_m z_2) = 0 \quad . \tag{6.14}$$

Further, the eigenfunctions f_m satisfy the orthogonality conditions

$$\int_{z_1}^{z_2} z\, f_m\, dz = 0 \quad , \quad \int_{z_1}^{z_2} z\, f_m\, f_n\, dz = 0 \quad \text{for} \quad (m \neq n) \quad . \tag{6.15a,b}$$

Substituting (6.11) into (6.9), multiplying the result by z and integrating, multiplying the result by zf_n and integrating, and using the orthogonality condition (6.15) and the initial conditions (6.10a,b), we conclude that

$$A_o(\tau) = C_o\tau \quad , \quad C_o = \left[1 + \frac{\beta}{(1 + \beta^2)^{1/2}}\left(\frac{1}{z_1 + z_2}\right)\right] \quad , \qquad (6.16a,b)$$

$$A_m(\tau) = \frac{(1 + \beta^2)C_m}{\alpha_m^2}\left[1 - \exp\left\{-\left(\frac{\alpha_m^2}{1 + \beta^2}\right)\tau\right\}\right] \quad , \qquad (6.16c)$$

$$C_m = \frac{\dfrac{\beta}{2(1 + \beta^2)^{1/2}} \displaystyle\int_{z_1}^{z_2} f_m \, dz}{\displaystyle\int_{z_1}^{z_2} z \, f_m^2 \, dz} \quad , \qquad (6.16d)$$

$$B_o(\tau) = \frac{1}{2} C_o[1 - \exp(-\pi^2\tau)] \quad , \qquad (6.16e)$$

$$B_m(\tau) = \frac{6 \, C_m}{\left(12 + \dfrac{\alpha_m^2}{1 + \beta^2}\right)}\left[1 - \exp\left\{-\frac{\pi^2}{12}\left(12 + \frac{\alpha_m^2}{1 + \beta^2}\right)\tau\right\}\right] \quad . \qquad (6.16f)$$

For later reference, we observe that if the dependence on z is neglected in (6.11), then (6.11) has the same form as the solution (4.8). This means that we would be essentially modeling the conical shell as an "equivalent" circular cylindrical shell with "mean" radius $R/h = (z_1 + z_2)(1 + \beta^2)^{1/2}/2\beta$. By considering a specific example, it will be shown that making this kind of engineering approximation introduces significant errors at the tip of the conical shell.

Consider the specific example of the conical shell drawn in Figure D.1, which has a solid tip. For this shell we specify

$$z_1 = \frac{\beta}{2(1 + \beta^2)^{1/2}} \quad , \quad z_2 = 15 \quad , \quad \beta = 3.23 \quad . \qquad (6.17a,b,c)$$

The minimum value z_1 of z given by (6.17a) is specified by requiring the inner surface of the shell to just make contact at the shell's tip. Using (6.17) we have solved for the first twenty eigenvalues and eigenfunctions and have plotted the solution (6.11) in Figures D.11 and D.12 by normalizing the results by the first terms in the solutions. Figure D.11 shows plots of $\bar{\theta}/A_0(\tau)$ verses z for various values of τ, and Figure D.12 shows plots of $\bar{\phi}/B_0(\tau)$ verses z for two values of τ. The slight waviness in these curves is caused by the fact that we have approximated solutions (6.11a,b) using finite series.

From Figure D.11, we observe that for long times the average temperature is relatively uniform over the shell. This is because the equivalent-cylinder solution $A_0(\tau)$ dominates for long times. However, for short times the value of $\bar{\theta}$ at the tip is about 65% greater than that predicted by the equivalent-cylinder solution. This result can be explained by observing from (4.8a) that a thick cylinder heats up faster than a thin cylinder. Thus, we would expect the tip of the conical shell, which is thick, to heat up faster than its base, which is thin. From Figure D.12, we observe that the distribution of the average temperature gradient is nearly constant with time. Also, the value of $\bar{\phi}$ near the tip is nearly 65% greater than the value predicted by the equivalent-cylinder solution.

To exhibit the temporal dependence of this solution more clearly, we have plotted $A_0(\tau)$ and $B_0(\tau)$ in Figure D.13. From this figure, we observe that $B_0(\tau)$ reaches its maximum value in a relatively short time. Recalling from [2] that the average temperature gradient is related to the thermal bending moment in the shell, this means that the bending in a conical shell under this load will be quite severe at its tip and the full effect of the load will be felt in a relatively short time. Consequently, the tip of the conical shell should be particularly vulnerable to this type of thermal load.

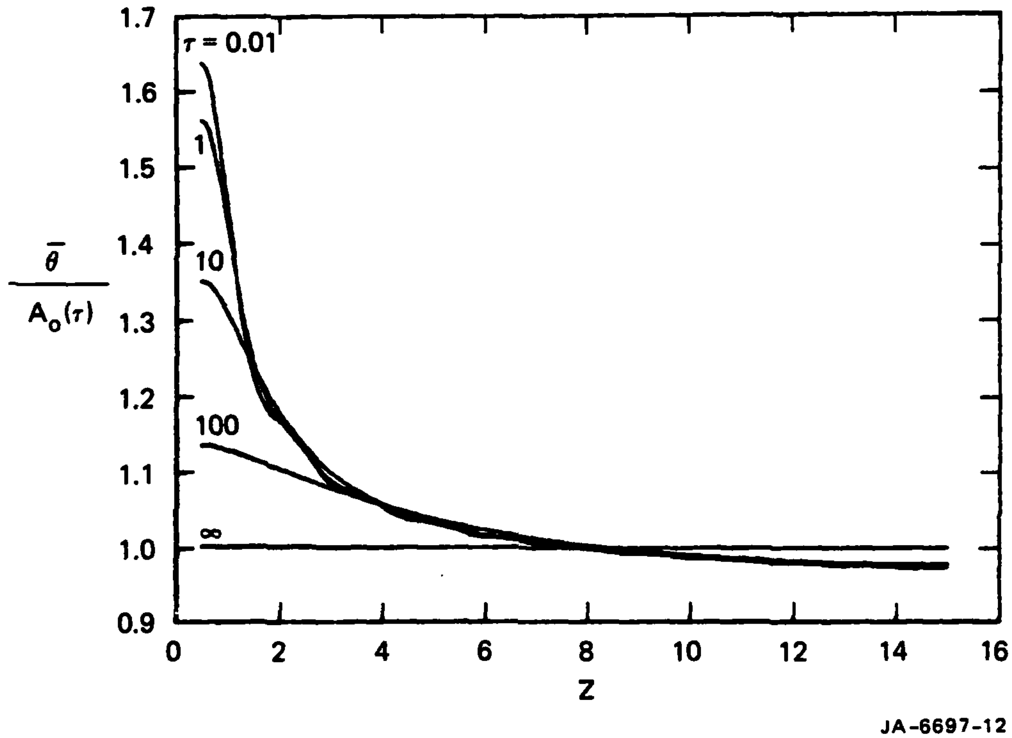

FIGURE D.11 NORMALIZED AVERAGE TEMPERATURE IN A CONICAL SHELL OF THICKNESS h.

The numbers on the curves are values of $\tau = Kt/\rho_o^* ch^2$.

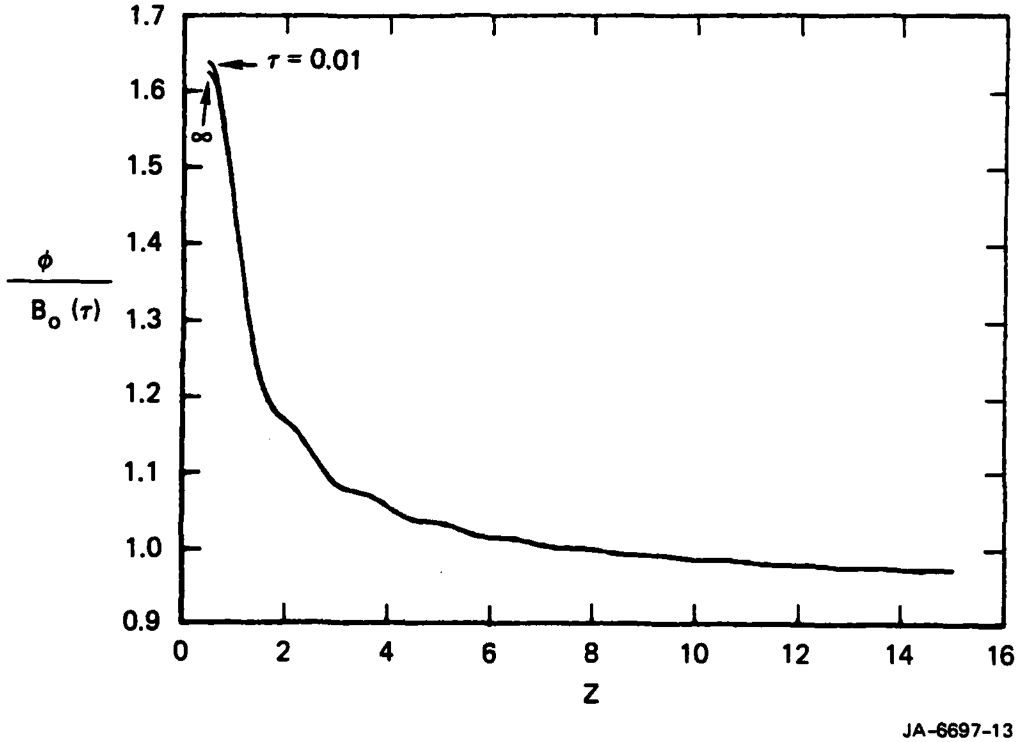

FIGURE D.12 NORMALIZED AVERAGE TEMPERATURE GRADIENT IN A CONICAL SHELL OF THICKNESS h

The numbers on the curves are values of $\tau = Kt/\rho_o^* ch^2$.

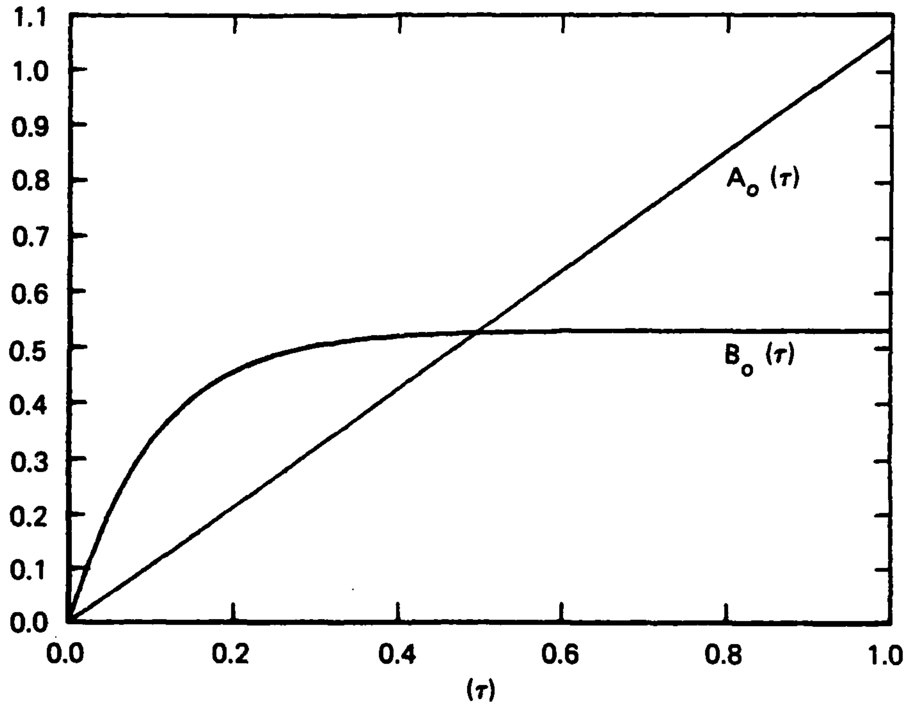

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FIGURE D.13 VALUES OF THE FUNCTIONS $A_o(\tau)$ AND $B_o(\tau)$ ASSOCIATED WITH THE CONICAL SHELL SOLUTION

7. Summary

In this paper, we have focused attention on analyzing heat conduction in a rigid conical shell (Figure D.1). The conical shell is particularly interesting because it has a converging geometry, so that the shell near its tip is necessarily "thick" even though the shell near its base may be "thin." Further, the heat conduction equation is not separable for the conical geometry, and hence it is exceedingly difficult to obtain exact solutions. We have chosen to model the shell with the theory of a Cosserat surface to determine the average (through-the-thickness) temperature and temperature gradient in the shell directly.

A number of problems of plates, circular cylindrical shells, and spherical shells are considered and the solutions are compared with exact solutions to develop confidence in the Cosserat theory. Within the context of this theory, it is usually assumed that constitutive equations for shells have the same form as those for plates. Here, it is shown that to predict relatively accurate results in the thick-shell limit, it is necessary to generalize these constitutive equations to include certain geometrical features of the shell. The generalized constitutive equations are developed here in a consistent manner and tested in the thick-shell limit. The tests include problems where the temperature fields θ and ϕ are functions of time only so that their Laplacian vanishes, as well as problems where they are functions of space only and their Laplacian does not vanish. In all cases, satisfactory results are predicted even in the thick-shell limit.

Finally, a problem of transient heat conduction in a conical shell, which does not have an exact solution, is solved analytically using the Cosserat theory. It is shown that both the average temperature and temperature gradient have values near the tip that are about 65% greater than those predicted by an approximate equivalent-cylinder solution. Also, it is shown that the thermal bending moment produced by the average temperature gradient is quite severe near the tip and it attains its maximum value in a relatively short time.

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